The Butterfly and the Tornado

Chaos theory and climate change

Everything is mathematical

The Butterfly and the Tornado

The Butterfly and the Tornado

Chaos theory and climate change

Carlos Madrid

Everything is mathematical

For Gustavo Bueno

Published by RBA Coleccionables, S.A.
c/o Hothouse Developments Ltd
91 Brick Lane, London, E1 6QL

Localisation: Windmill Books Ltd.
Photographs: iStockphoto

ISSN: 2050-649X

Printed in Spain

Contents

Preface

Can the flap of a butterfly's wings in Brazil set off a tornado in Texas? Of course it can, and if you have already read about chaos, no doubt you will know this. However, perhaps what you don't know is the answer to the opposite question: can the flap of the same butterfly's wings in Brazil prevent a tornado over Singapore?

If the latter question has taken you by surprise and you want to find out the answer, you are reading the right book. The majority of books dealing with chaos theory and its connection to meteorology and climatology only answer the first question, leaving the second to one side. However, this book seeks to tackle both – it shows the two faces of chaos. (Of course, the answer to the second question is also yes.)

The butterfly after which this book is titled is not so defenceless when it comes to the tornado as we might at first believe. Lorenz's butterfly has become the symbol of chaos theory, just like Schrödinger's cat has become the symbol of quantum mechanics. Unfortunately, Lorenz's butterfly is just as difficult to tame as Schrödinger's cat, since chaos theory and quantum mechanics are, without doubt, the two most serious ruptures in the scientific ideals of predictability and determinism. Especially in our case, bearing in mind that one of the most disturbing properties of chaos is its ubiquity. The Solar System, the weather, the climate, populations and epidemics, turbulence, the dripping of a tap, certain chemical reactions, the smoke of a cigarette, the beating of a heart, brain signals and financial markets are just some examples of the many chaotic systems. What is really amazing, however, is not that certain complex systems are chaotic, but that this is also the case for extremely simple systems, such as a double pendulum.

This book deals with chaos, or rather, the erratic and unpredictable behaviour of certain dynamic systems and their relationship to a highly topical problem: climate change. Chaotic behaviour arises when there is a sensibility to initial conditions, a mechanism that is referred to as the 'butterfly effect', and we experience it on a daily basis in weather forecasts, and also, as we shall come to see, in climate predictions. Few issues related to science arouse such interest as climate change. However, in order to approach the issue in the same way as scientists, we need to be able to distinguish between the alarmist media, which sometimes takes over, and mathematics, which defines the real behaviour of the climate system.

After considering the revolutionary consequences of chaos (that will sour the face of a great philosopher), the first two chapters will provide an overview of the

birth and history of chaos theory. In the third chapter we shall explain the main concepts related to chaos, including its most modern, interdisciplinary applications. Finally, in the last two chapters we shall illustrate how these methods and concepts are applied to the problem of climate change, which we shall also attempt to set out in a general manner that can be understood by all.

Writing an entertaining yet nonetheless in-depth book on chaos theory is no mean feat. The same is true of climate change. Yet, to do the same for both topics at once is not just doubly difficult, but quadruply so (at the very least). It is a matter of non-linearity. Jokes aside, if the reader reaches the last page of this book having understood the reason behind this play on words, we shall be satisfied, since it means that they will have penetrated the nucleus of chaos theory and the problems it tackles. I am, of course, referring to the non-linear dynamic in which the sum of two things can often be explosive...

Combining mathematics and popular science has represented a quantum leap that has revolutionised my point of view. Little by little, the uncertainty has disappeared and both types of knowledge, scientific and human, have appeared as complementary doubles. At any rate, this perturbation of my life would not have been possible without the modification of the initial conditions under which my school and university teachers worked, setting me out on my chaotic steps towards the strange attractor of mathematics and its history. There is no reason not to express my gratitude to the generosity of the many people who have provided encouragement to write this book – from Elena, my mother, the Casados and Arribes, all the way to Javier Fresán and my friends and colleagues at the institute and the university, who do not wish to read the book but put up with me as I wrote it.

There remains only one thing to do, encourage you to enter, read and be seduced by chaos.

Chapter 1

The Prehistory of Chaos Theory

In fact, the greater the science, the greater the mystery.
Vladimir Nabokov

Immanuel Kant (1724–1804), the great philosopher, globally renowned, returns from his daily walk. His servant follows at a safe distance, trying not to intrude on the thoughts of his master. Kant walks quickly, although with short steps, and always in the same place at the same time every day. The inhabitants of Königsberg are by now accustomed to taking advantage of the mathematical punctuality of their eminent neighbour to set their watches. Herr Kant is as precise in his step as the Earth in its rotation round the Sun. Today, however, after walking across the garden and reaching the doorstep, the author of *Critique of Pure Reason* stops. He has paused to contemplate a plant that has grown during the recent rains. It is a fern. Among its green leaves, he can make out an insect that climbs clumsily up the trunk. It is a delicate butterfly. The genius strokes it, runs his hand over one of the wet leaves of the fern and smiles, admiring the geometric perfection of its shape. He mutters something to himself, looks at the sky above, and enters the house.

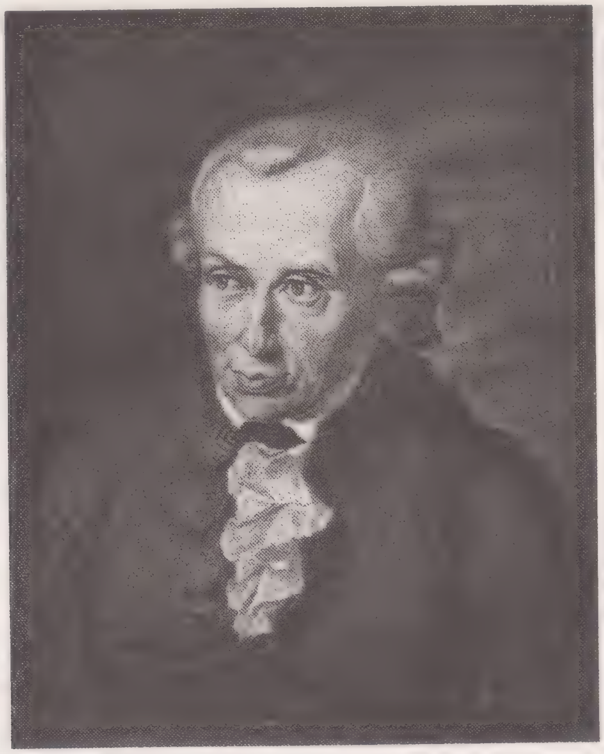
Minutes later, sitting at his desk beside the fire, he takes his pen, dips it into the inkwell and begins to write.

If Kant were to look up…

In his *Critique of Judgement*, after asking whether it is nature itself or the mathematician that is responsible for the introduction of mathematics to natural philosophy, Immanuel Kant writes with respect to the prevailing mechanisms in nature:

> "and we can say boldly it is alike certain that it is absurd for men to make any such attempt or to hope that another Newton will arise in the future,

who shall make comprehensible by us the production of a blade of grass according to natural laws which no design has ordered. We must absolutely deny this insight to men."



Portrait of Immanuel Kant.
"In the earliest times of which history affords us any record, mathematics had already entered on the sure course of science."

However, this ambitious statement is now obsolete, since, if we allow the comparison, the time has now come for a second Newton, a Newton of the fronds of ferns. His name? Michael Barnsley, an English mathematician and expert on one of the most interesting products of chaos theory: fractals. Fractal geometry, as we shall see throughout the course of this book, is the inseparable companion of chaos theory.

Barnsley discovered, by means of a simple 'chaos game' that it is possible to make the leaves of a fern, broccoli, etc. appear, as if by magic. A chaos game consists of nothing more than drawing points randomly until the series of points gives us, at its limit, a recognisable image. In short, using a random law – in the words of Kant, a law that is not ordered by intention – and with the help of a computer, we can cause the leaf of a plant to 'sprout'. Simplifying, it is enough to proceed as follows. We choose a point (which is not the central point of the screen) and simulate the tossing

of a coin. If it comes up heads, we mark a new point exactly six units north-west of the previous point, and, if it comes up tails, we mark it by moving 25% towards the central point with respect to the previous point. Clearly, this procedure can be iterated as many times as we wish. To begin with, the distribution of the points that are drawn appear randomly, by chance. However, mysteriously, after a thousand iterations, a specific shape begins to emerge – the faint shape of a fern leaf. It is as if order were arising from chaos in the shape of a fractal set. This is the Barnsley fern.

'Spontaneous' generation of the Barnsley fern.

AN EXTRACT FROM ALEJO CARPENTIER'S NOVEL *EXPLOSION IN A CATHEDRAL*

Beholding a snail – just one – thought Esteban of the presence of the Spiral over millennia and millennia, before the everyday gaze of fishing people, still unable to understand it or perceive the reality of its presence. He meditated on the poem of the sea urchin, the helix of the razor clam, the rays of the scallop, astounded by the Science of Shapes that had unfolded over so many years in the presence of humankind, which still lacked eyes to conceive it. What could there be around me that is already defined, written, present, *yet* which cannot yet be understood? What sign, what message, what warning lies in the curls of the chicory, the alphabet of mosses, the geometry of the rose garden? To behold a snail. Just one. Te Deum.

It is impossible to know what the great philosopher of Königsberg would say if he saw the surprising number of natural systems with a chaotic dynamic, together with all that this entails. Put another way it is a random or stochastic behaviour (in Greek *stochastikos* meant 'a good shot') within a strict determinism. Many erratic movements, without apparent order, actually conform to fixed rules that are not the result of chance. Chaos and fractals constitute a new way of exploring the Universe.

The genesis of chaos theory

Chaos is everywhere. In the cinema, we can find it in films such as *Chaos, The Butterfly Effect* and *Jurassic Park*. In literature it is to be found in novels such as *The*

SCRIPT FROM THE FILM *JURASSIC PARK* (STEVEN SPIELBERG, 1993) BASED ON THE NOVEL BY MICHAEL CRICHTON

"The Tyrannosaurus does not obey a fixed system and the hours of the park, it is the essence of chaos."

"I don't understand the part about chaos. What does it mean?"

"It refers to the unpredictability of complex systems. It is summarised by the butterfly effect. A butterfly flaps its wings in Peking and in New York it rains instead of being sunny. Am I going too fast? Give me that glass of water. Let's see. The car does not stop jumping, but it doesn't matter, it's just an example. Stretch out your hand, like a hieroglyph. Imagine that a drop of water falls onto your hand. Which way will it run? Towards your thumb? Towards the other side?"

"I don't know... Towards my thumb!"

"Good, now keep your hand still, don't move it, I'm going to do the same thing, in the same place. Which way will it go?"

"I don't know... The same way?"

"It has changed! Why? As a result of microscopic variations (the direction of the hairs on your hand, the quantity of blood running through the vessels, the microscopic imperfections of your skin... that are never repeated and have a significant effect on the result. In other words... unpredictability. Look, nobody would have been able to predict that Doctor Grant would suddenly jump from a moving car. I have another example. Here I am talking to myself. That, that is chaos theory."

Painter of Battles by the Spanish author Arturo Pérez-Reverte, in which a photograph taken at random completely changes the life of a Croatian soldier; and in stories such as *The Sound of Thunder* by Ray Bradbury, in which the death of a prehistoric butterfly alters the result of a presidential election in the United States; and *Il crollo della Baliverna*, by Dino Buzzati, in which a leisurely climb over a run-down wall leads to an unexpected ending.

But what is chaos? The majority of dictionaries provide various definitions of the term. Here are three. The first and the second refer, respectively, to its use in classical Greece and its colloquial meaning:

1. Amorphous or indefinite state that came prior to the ordering of the cosmos.

2. Confusion, disorder.

However, the third definition gives us its meaning in mathematics and physics:

3. Apparently erratic and unpredictable behaviour of certain dynamic systems, despite their mathematical formulation being primarily deterministic.

Naturally, this book deals with the third, scientific definition of chaos, although it also considers how mathematical chaos has found its place in the collective imagination thanks to technological applications in fields such as physics, biology, medicine and neuroscience. The mechanisms at work in our world, from the human brain to the climate, are impregnated with chaos.

In this first chapter and the following ones, we will retrace the history of the mathematical theory of chaos, which leads us from the days of Newton and the Scientific Revolution to the present day. However, it was during the 19th and 20th centuries when a series of open problems in celestial mechanics related to the stability of the Solar System (e.g, Would the Moon collide with the Earth? Would an asteroid crash into our planet, causing an end to human life?) required a brilliant mind to shed light on them. Cue Henri Poincaré.

In this chapter and the following we are going to work with an intuitive idea of chaos that lies extremely close to that used in mechanics, since this was the first discipline to describe the strange movements that are now associated with chaotic systems. In the third chapter, we shall attempt to formalise things a little

more, providing a precise definition of the 'butterfly effect' of chaos mentioned in the introduction and which has already come up in our review of literature and the cinema.

However, let us take things step by step and begin at the beginning. So-called chaos theory was born of certain mathematicians with an interest in the relationship between dynamic systems (systems that evolved over the course of time) and geometry, such as the aforementioned Henri Poincaré and Stephen Smale; certain physicists from fields as varied as meteorology and astronomy, such as Edward Lorenz and Michel Hénon and certain biologists who worked on population growth, such as Robert May. However, to this long list, we should also add figures such as James Yorke, David Ruelle, Mitchell Feigenbaum, Michael Barnsley and many others on account of their own multidisciplinary scientific achievements. However, how did it all begin? What is the true history of chaos?

We are about to embark on a journey to the sources of chaos theory, in which we will travel the three rivers that flow out into the sea of dynamic systems: Newtonian mechanics, the analytical mechanics of Laplace and, finally, the general theory dreamt up by Poincaré – a figure who stands out in his own right as the main character of this chapter.

From Newton to Laplace, via Leibniz

The attempt to understand the planetary trajectories observed by Kepler led Newton to model them mathematically, following in the footsteps of Galileo. Thus did he provide a mathematical formulation of his laws on physical magnitudes and their speeds of change, such as the distance covered by a moving body and its speed, or the speed of the moving body and its acceleration, to give just two examples. The physical laws that described dynamic systems were thus expressed using differential equations, in which the differentials were measurements of the rates of change.

A differential equation is an equation whose primary unknown is the rate of change of a magnitude, or in other words, its 'differential' or 'derivative'. Both the differential and the derivative of a function represent how its value changes, in other words whether it increases, decreases or remains constant. Acceleration, to continue our example, measures the changes in the speed of the moving body, which is the quotient of the differentials of speed and time; in other words, it is

the derivative of the speed with respect to time. Consequently, it expresses the variation of speed over time.

However, just as with algebraic equations (those from everyday life), it is not always easy to solve differential equations. In fact, it is almost never easy to solve them. In this respect, the subsequent analytic mechanics represented progress with respect to Newtonian mechanics, since by bringing mechanics towards analysis and moving it away from geometry, the study of physical phenomena and the discovery of differential equations that governed them became synonymous. Hence, after Newton's discovery of the famous differential equation – force equals mass times acceleration, which governs the motion of systems of points and rigid solids – the Swiss mathematician Leonhard Euler (1707–1783) devised a system of differential equations that describes the motion of continuous media, such as water, air and other non-viscous fluids.

Later, the mathematician and physicist Joseph Louis Lagrange (1736–1813) focused his attention on sound waves, or in other words, acoustic equations. Finally, Jean-Baptiste Fourier (1768–1830) focused on the flow of heat, proposing an equation to describe it. In Fourier's opinion, mathematical analysis was as extensive as nature itself.

Between the 17th and 19th centuries, physicists extended their mathematical dominion over the world as they proposed new differential equations for the study of phenomena originating from any field, such as the Navier–Stokes equations for viscous fluids and Maxwell's equations for electromagnetics. Hence, all aspects of nature – solids, fluids, sound, heat, light, electricity – came to be modelled using differential equations. However, it was one thing to find the equations for the phenomenon in question and something else to solve them.

In principle, there are two types of differential equations: linear equations and non-linear equations. A differential equation is linear if the sum of two solutions gives another solution. Furthermore, in a linear equation neither the unknown function nor its derivative are raised to a power other than zero or one. Linear differential equations model phenomena in which the effect of a summation of causes is the sum of the effects of each of the individual causes considered separately. On the other hand, non-linear phenomena and equations do not exhibit this sort of proportionality between cause and effect, such that the conjunction of two different causes may be explosive. As we shall see, such non-linearity always lies behind chaos.

NEWTON AND THE FIRST DIFFERENTIAL EQUATION

The most famous differential equation is, without a doubt, that credited to Newton: "Force equals mass times acceleration." Symbolically: $F = m \cdot a$, where $a = \frac{dv}{dt}$ (acceleration is the quotient of the differentials of speed and time, or in other words, the derivative of speed with respect to time). Let us consider two more examples to make things clearer:

$$\frac{dy}{dx} + y = 0$$

is a linear differential equation, however

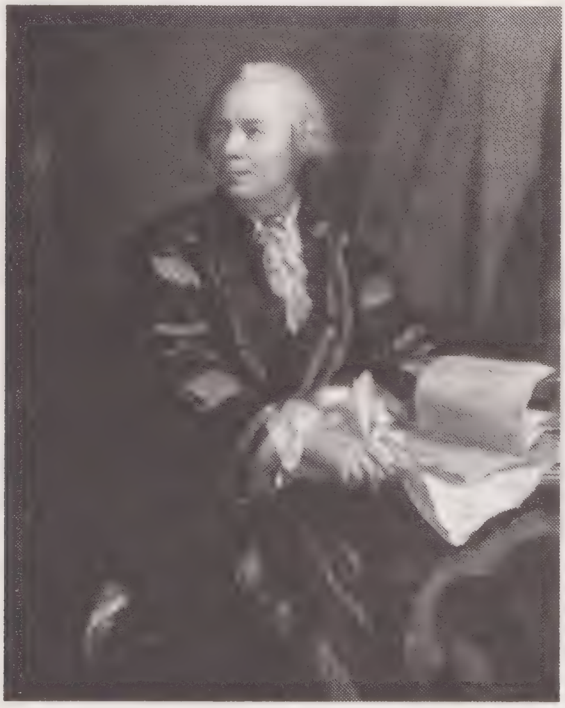
$$\left(\frac{dy}{dx}\right)^m + y^n = 0$$

for m and n not equal to 0 or 1, is a non-linear differential equation due to the fact that both the unknown function and its derivatives are raised to powers other than 0 and 1.

The theory of linear differential equations was soon developed completely. However, the same thing did not occur with the twin theory of non-linear differential equations. Non-linear problems, such as the pendulum equation, were solved by 'linearising' them or, in other words, removing the awkward terms from the equation. This meant that for a given non-linear differential equation a similar linear differential equation was solved, and the solutions that were obtained were used to approximate solutions to the non-linear equation. This was the so-called 'perturbation method'. However, the technique soon came to reveal its limitations, since it did not work for multiple cases. It would be a long time before non-linear equations received the same level of attention as linear equations.

One of the non-linear problems that has been a thorn in the side of physicists and mathematicians since the 17th century lies within the realm of celestial mechanics. It is the modelling of the Solar System, the n-body problem, which can be stated simply: given n bodies with different masses under mutual gravitational attraction, it is necessary to determine the motion of each of these in space. Although the statement of the problem is apparently extremely simple, its solution is not always quite so easy. Newton provided a geometric solution to the two-body problem for two spheres moving under mutual gravitational attraction in *Philosophiæ naturalis principia mathematica* (*Mathematical Principles of Natural Philosophy*), and in 1734 Daniel

Bernoulli (1700–1782) provided an analytic solution in a paper that received a prize from the French Academie des Sciences. However it was not until 1744 that Euler provided a fully detailed solution in his work *Theoria motuum planetarum et cometarum*.



Portrait of Euler.
"Read Euler, read Euler, he is the master of us all" (Laplace).

THE NON-LINEAR EQUATION OF THE PENDULUM

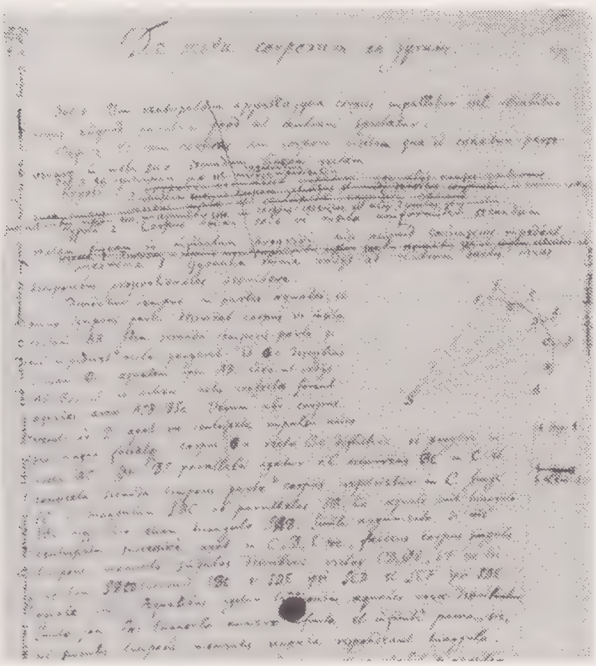
If θ represents the angle of inclination of the pendulum with respect to the vertical, $\frac{d^2\theta}{dt^2} + \sin\theta = 0$ is the non-linear differential equation for the pendulum. For small oscillations, the equation can be 'linearised' by approximating the trigonometric function $\sin\theta$ by θ. The resulting equation $\frac{d^2\theta}{dt^2} + \theta = 0$ can now be easily solved (it is a second-order differential equation since it contains a second derivative, but note that neither the second derivative nor θ are raised to an exponent greater than one).

Another example of a non-linear differential equation is the following: $m\frac{dv}{dt} - v^2 = mg$, where g is the acceleration of gravity (9.8 m/s^2), that describes the motion of a projectile in a medium whose resistance is proportional to the speed squared (v^2). That is precisely the term that makes the equation non-linear).

After solving the n-body problem for $n = 2$, the physicists and mathematicians of the 18th and 19th centuries began to tackle the same problem for $n = 3$, required to understand the motion of the system of the Sun, the Earth and the Moon. At this point, two parallel research programmes began. The first saw the search for general solutions, approximated using the perturbation method, whereas the second sought exact, specific solutions. Hence, for example, Lagrange solved the three-body problem by restricting it to the system formed by the Sun, Jupiter and the asteroid Achilles. Lagrange's most famous work, his *Mécanique analytique* (*Analytical Mechanics*) crowned Newton's work. Although his idol was Archimedes, Lagrange complained on a certain occasion that Newton had been a lucky man, since there was just one single Universe and he had discovered its mathematical laws.

Simultaneously, a matter extremely closely related to the n-body problem reared its head – the stability of the Solar System. At that time this was a system made up of just seven bodies, and the solution directly depends on that of the n-body problem. Newton knew that an exact solution to the two-body problem could be found for any moment in time. However, this was not the case when a third body joined the interaction. Although weak in comparison to the Sun's force of attraction, the forces between the planets were by no means insignificant, since in the long term they could cause a planet to deviate from its orbit and even, in an extreme case, expel it from the Solar System. The interplanetary forces could shatter the beautiful Keplerian ellipses, making it impossible to predict the behaviour of the Solar System in the long-term. In fact, in his work *De motu corporum in gyrum* (1684), Newton stated that the planets did not even move in perfect ellipses or repeat the same orbit twice. Furthermore, he recognised that defining this motion for the totality of the future lay far beyond the powers of the human intellect.

Hence, the following urgent question remains: is the Solar System


De motu corporum in gyrum.

Page from the manuscript of Newton's De motu corporum in gyrum.

stable or unstable? Will each heavenly body stay in its orbit, or will a deviation occur in the future? For Newton, if the Solar System was losing its balance, it required a drastic solution – the hand of God would be required periodically to push each planet from inside its ellipse to re-establish the harmony from time to time. Faced with Newton, Leibniz maintained that the Creator could not be such a clumsy clockmaker. The Newtonian *deus ex machina* invoked the wrath of the horrified pious Leibniz.

Many decades later, Pierre-Simon Laplace (1749–1827), the great mathematical physicist who rose to the ranks of Minister of the Interior under Napoleon (although he was dismissed after just a few months, since "he brings infinitesimal calculus", Napoleon said with some irritation, "to bear on all aspects of life"), thought to

THE LEIBNIZ–CLARKE DISPUTE

Between 1715 and 1716, the philosopher, mathematician, lawyer and ambassador, to list just some of his many professions, Gottfried Leibniz (1646–1716) sustained a correspondence with Samuel Clarke (1675–1729), an Anglican cleric and follower of Newton. Their debate, not exempt from theological matters, dealt with the consequences of Newtonian mechanics for Orthodox Christianity. Leibniz had already entered into a more crude epistolary discussion with Newton himself on the priority of calculus (differential and integral), which saw both mutually accuse each other of plagiarism. In this case, among other things, Leibniz challenged Newton's findings with respect to the three-body problem and the stability of the Solar System. The alleged perfection of God meant that this world was necessarily the best of all possible worlds and, hence, it was absurd that God would need to intervene from time to time to synchronise the Universe's clock. Why would God have created a world in which the planets could escape from their orbits? In Leibniz' opinion, Newton's theories undervalued the divine power. Indeed, in *Opticks*, Newton states: "But by reason of the Tenacity of Fluids, and Attrition of their Parts, and the Weakness of Elasticity in Solids, Motion is much more apt to be lost than got, and is always upon the Decay." To which Leibniz asked: "Is the machinery of God capable of falling into such disorder that He is required to repair it, as if he were an artisan?" To avoid stooping to this level, Newton answered via Clarke… and the Leibniz–Newton dispute culminated in a schism of British mathematics from that of continental Europe for a long time. And vice versa: the French, for example, followed Descartes and his antediluvian theory of vortices until Voltaire returned from England in 1727 supporting Newton's theory of gravity.

explain the anomalies in the orbits of Saturn and Jupiter, which were of such concern to Newton, as mere perturbations that depended solely on the law of gravity and that tended to compensate each other with the passing of time. Jupiter was subject to an apparent acceleration, whereas Saturn appeared to be gradually slowing, such that if these movements continued indefinitely, the former would escape from the Solar System. Laplace proved that the acceleration of Jupiter and the deceleration of Saturn were caused by small effects, of a secondary order, due to the relative position of both planets with respect to the Sun. The Solar System was self-regulating. Almost one hundred years later, it appeared that Leibniz the optimist had won out over the gloomy Newton. Hence, when he presented his *Traité de mécanique céleste* (*Treatise on the Celestial Mechanics*) to Napoleon, his patron pointed out to him that he had been unable to find God's name in any of the volumes of his monumental work. Laplace answered that God was not a necessary hypothesis in his system of the world – a system he believed was fully predetermined and stable. In a passage extracted from the pages at the start of his *Philosophical Essay on Probabilities* (1814):

> "We ought then to regard the present state of the Universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it an intelligence sufficiently vast to submit these data to analysis, it would embrace in the same formula the movements of the greatest bodies of the Universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past would be present in its eyes. The human mind offers, in the perfection which it has been able to give to astronomy, a feeble idea of this intelligence. Its discoveries in mechanics and geometry, added to that of universal gravity, have enabled it to comprehend in the same analytical expressions the past and future states of the system of the world."

However, Laplace's response was light-years from being correct. In his equations of the Sun-Jupiter-Saturn system (three-body problem), Laplace overlooked a mathematical term he regarded as extremely small but which in fact was able to grow rapidly and without limit until it destablilised the Solar System. While Lagrange was a mathematician who took great care in his writings, Laplace was like a fox who erased his tracks with his own tail. He frequently forgot to acknowledge the

source of his results, giving the impression that they were his own, and often solved the mathematical problems he encountered in his physics research in passing. The American astronomer who translated the *Treatise on Celestial Mechanics* into English stated that every time he encountered the phrase "it can be easily seen that..." He knew that he would be faced with hours of hard work to fill the gaps in the text.

Portrait of Laplace (1749–1827), 'the Newton of Revolutionary France'.

There were many 19th-century physicists and mathematicians who gave themselves over to providing a full answer to the three-body problem and the stability of the Solar System, and more than 800 works can be counted on the topic between the time of Newton and 1900. However, the mathematicians kept awake by the problem included a key figure in the configuration of chaos theory – the brilliant Henri Poincaré (1854–1912).

King Oscar's competition

Ever since he was a child, Poincaré was extremely adept at mathematics, despite being clumsy and distracted when it came to other things. In fact, he is regarded as the last generalist mathematician. In contrast to modern specialists, a universal curiosity

led him to investigate topics ranging from analysis, differential equations, groups, topology, celestial mechanics and mathematical physics, together with philosophy, teaching and popular science. And of course, he was the first mathematician to come face to face with chaos, thanks to his study of the three-body problem.

Jules Henri Poincaré at the age of 36.
"Thought is only a flash between two long nights, but this flash is everything."

Poincaré's illustrious report on the problem was published in 1890, at the age of just 36, although the history of the publication dates much further back. A number of years before, in 1885, European mathematicians received news of an important international mathematical competition organised under the patronage of Oscar II, King of Sweden and Norway, who had an enthusiasm for mathematics as a result of some courses he studied at university. As part of an international competition, he offered a prize to the mathematician who was finally able to solve the three-body problem and hence open the way to studying the Solar System.

In fact, in 1884, Gösta Mittag-Leffler (1846–1927), professor of pure mathematics at the University of Stockholm, suggested to Oscar II the idea of a mathematical competition to commemorate the 60th birthday of the monarch five years later

on 21 January 1889. It should be noted that competitions of this nature were not uncommon at the time, and while the prizes were not large in financial terms, they represented considerable prestige for their winners, comparable with the future Nobel Prizes. Mittag-Leffler also wanted to associate the competition with the journal *Acta Mathematica*, founded shortly beforehand thanks to the king's invaluable support, in order to attract important articles.

The nomination of the organising committee and the panel of judges was extremely problematic. The professor wished to avoid disputes and rivalries between the members, as well as future accusations of bias. So he decided on a panel of figures close to him – his old mentors Charles Hermite and Karl Weierstrass, as representatives of the French and German schools – and Sofia Kovalévskaya, a brilliant student of Mittag-Leffler and Weierstrass, who died an untimely death from influenza at the age of 41, depriving mathematics of a bright talent.

With the help of Mittag-Leffler, the committee drew up four questions, one of which required the *n*-body problem to be solved:

> "Given a system of arbitrarily many mass points that attract each according to Newton's law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly.
>
> "This problem, whose solution would considerably extend our understanding of the Solar System, seems capable of solution using analytic methods now at our disposal; we can at least suppose as much, since Lejeune Drichlet communicated shortly before his death to a geometer of his acquaintance [Leopold Kronecker] that he had discovered a method for integrating the differential equations of Mechanics, and that by applying this method, he had succeeded in demonstrating the stability of our planetary system in an absolutely rigorous manner. Unfortunately, we know nothing about this method, except that the theory of small oscillations would appear to have served as his point of departure for this discovery. We can nevertheless suppose, almost with certainty, that this method was based not on long and complicated calculations but on the development of a fundamental and simple idea that one could reasonably hope to recover through persevering and penetrating research.

"In the event that this problem remains unsolved at the close of the contest, the prize may also be awarded for a work in which some other problem of Mechanics is treated as indicated and solved completely."

When the advertisement was published in *Acta Mathematica*, Poincaré, who was just 31 years old, already had a reputation in the world of mathematics. In fact, the members of the panel had regarded him as a potential entrant, although it was difficult for him to make the decision. Mittag-Leffler was forced to send him a letter encouraging him to take part, to which Poincaré replied that he hoped to tackle the three-body problem, not to provide a solution, an endeavour which he believed to be almost impossible, but to at least obtain new relevant results worthy of being submitted to the competition.

Finally, as a result of the competition, Poincaré began in-depth research into many of the ideas on the qualitative theory of differential equations he developed between 1881 and 1885, and which had been brought together in a collection of four memoires, the first of which was *On Curves Defined by Differential Equations*. In these memoirs, Poincaré considered linear and non-linear differential equations, less from a quantitative perspective (searching for specific solutions) than a qualitative one (general study of the dynamic and stability), which led him to make reference to the nascent topology or *analysis situs* (the term used at the time). In contrast to Lagrange, who made great show of the fact that his *Analytical Mechanics* contained not one illustration, Poincaré was not afraid of making use of geometry once again.

In fact, aware of the impossibility of solving the majority of differential equations (especially non-linear equations for which the method of perturbations failed), or in other words, aware of the difficulty of integrating the equations and expressing their solutions as recognised functions, Poincaré undertook their geometric study. He began by considering the differential equation

$$\frac{dy}{dx}=\frac{P(x,y)}{Q(x,y)},$$

in which the derivative of y with respect to x is equal to the quotient of two arbitrary functions P and Q. And he considered the so-called 'singular' or 'critical' points, or rather the points (x, y) for which $P(x, y) = Q(x, y) = 0$. In other words, the points for which the derivative of y with respect to x was 0 divided by 0, which is the same as saying the that points that cannot be determined (recall that it is meaningless to divide by 0). This is why they are referred to as singular or critical points.

THE GEOMETRY OF RUBBER BANDS

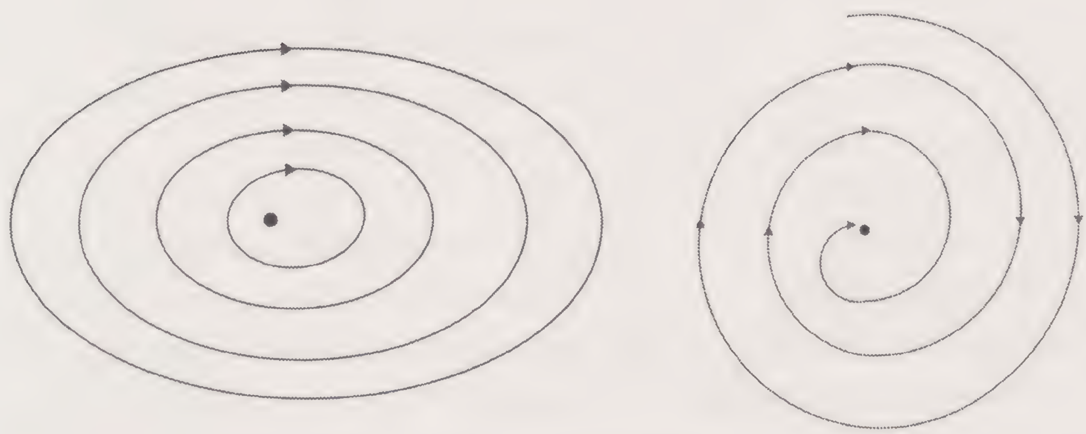
Topology is the mathematical science of geometric objects, concerning itself solely with their shape and position, and not quantitative properties, such as metric relationships. A classic example is underground maps, which preserve the information of the stations and interchanges between lines but distort the differences. Its main proponent was, of course, Poincaré, who popularised topolgy by presenting it as an extremely useful qualitative geometry. In his own words:

> "It is what is referred to as *analysis situs*. It is a whole body of theory that has attracted the attention of the greatest geometers, and from which, one after another, a series of important theorems has arisen. What distinguishes these theorems from ordinary geometry is that they are purely qualitative and remain valid if the shapes are copied by an unskilled draughtsman who grossly alters their proportions, replacing the straight lines by something akin to a curve."

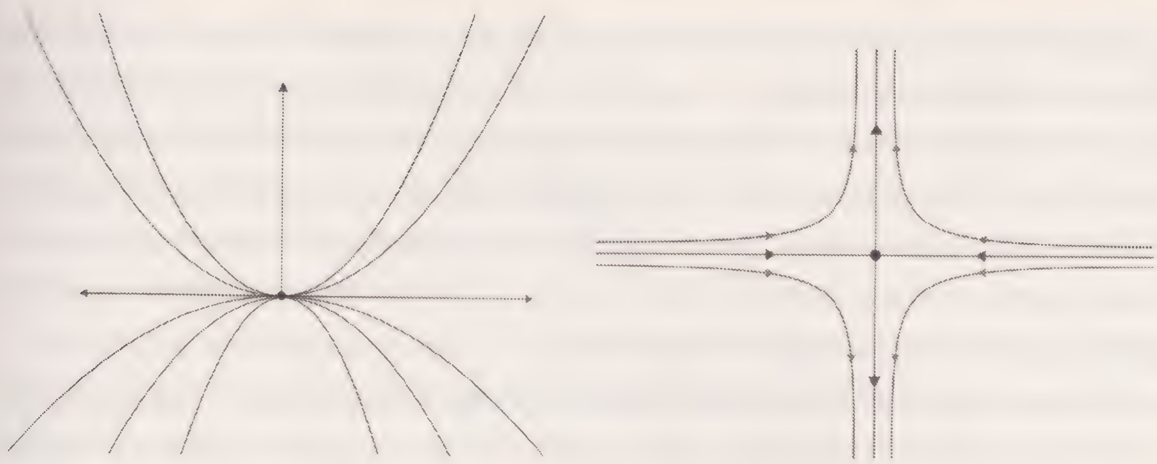
Poincaré defined topology as the geometry of rubber bands, since if the shapes were made of elastic rubber, it would be possible to deform many of them into others. For example, topologically speaking, a sphere and cube are indistinguishable, since it does not matter that one is smooth and the other has edges. There is a well-known saying that a topologist is nothing more than a mathematician who is unable to distinguish between a doughnut and a cup of tea. From their abstracted point of view, the only thing that matters is that both objects have a hole (the hole in the middle of the doughnut and the hole of the handle). We already know how to distinguish between a doughnut and an orange, depending on whether there is a hole, but how could we make this distinction if we were, hypothetically speaking, tiny inhabitants that lived on these surfaces? (The question is not trivial, since the sphere of the Earth on which we live appears flat.) One way of clearing up doubts is to study the Poincaré group for our space. Imagine that we have a dog we have tied to our house with an extremely long elastic lead and we allow it to roam about freely for a couple of days. When it returns, if we live on a doughnut, it is highly likely that it will return with the lead pulled tight, since it will have looped around the hole. On the other hand, if we live on an orange, it will return with the lead slack and we will be able to gather it in. Poincaré is also responsible for the famous conjecture that bears his name: "Is the three-dimensional sphere the only manifold of dimension three, such that any loop on it can be tightened to a point?" This generalised conjecture was successfully solved for four dimensions by Friedmann and then for more than four dimensions by Smale. However, the solution for three dimensions remained elusive until 2003 when the Russian mathematician Grigori Perelman announced he had found a proof.

Poincaré then went on to study them using topology. He observed how the curves defined by the differential equation behaved in their neighbourhood, since, at the end of the day, the solutions to the original differential equation are functions such that each makes it possible to draw a curve on the plane. Put more precisely, they make it possible to draw a curve on the so-called phase plane. The word 'phase' comes from the field of engineering and refers to the state or location of a given solution. Hence the phase plane gives us a portrait of the family of curves that represent the solution to the differential equation, often referred to as trajectories, or by analogy with the motion of planets, orbits.

Poincaré classified the singular or critical points into four groups: centres, sinks, sources and saddle points. The names come from the field of fluid mechanics, by analogy to the phase plane and their trajectories or orbits for a fluid that was spreading out or spilling over it. 'Centres' are points of equilibrium surrounded by periodic orbits; 'sinks' are stable points of equilibrium that attract the trajectories around them (they are like the drains of the phase plane's); 'sources' are unstable points of equilibrium, since they repel the trajectories around them (staying with the plumbing imagery, they are the taps from which the fluid on the phase plane emerges); and, finally, 'saddle points' are points of equilibrium that are both stable and unstable at the same time. They give the impression of two fluids colliding with each other. Trajectories that appear to intersect at the saddle point are referred to as 'separatrices'. Poincaré called the saddle points 'homoclinic' and the separatrices 'doubly asymptotic'. We shall see why at the end of the chapter.

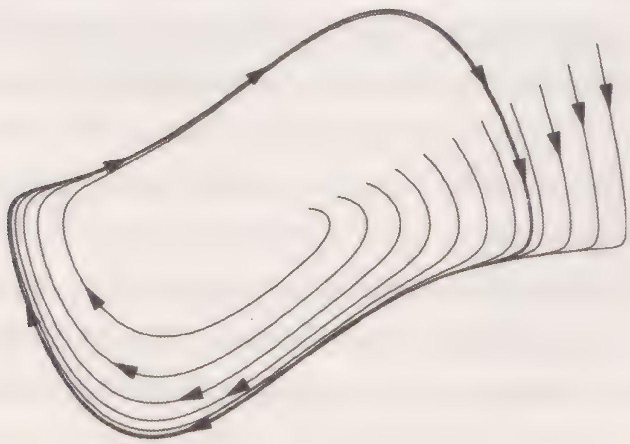


A centre (left) and a sink (right).



A source (left); a saddle point with two separatrices (right), which in this case are the two straight lines that cross the centre point.

In a subsequent theorem, now called the Poincaré–Bendixson theorem (after the Swedish mathematician who completed the proof), Poincaré showed that, together with the limit cycles (the closed curves that attract neighbouring orbits), this is the only type of dynamic process possible on the plane. There is no other sort of behaviour in two dimensions that is not 'moving close to' or 'moving away from' a singular point and, in its case, forming a periodic orbit (which defines a cycle, a closed curve). Since there is nothing more than centres, sinks, sources, saddle points and limit cycles in two dimensions, there are limited alternatives for the trajectories that represent the solution to the differential equation. These are repeatedly orbiting a centre or limit cycle, moving away from a source, passing close to a saddle point, or moving irrevocably towards a sink. The options can be counted on one hand.



Limit cycle of the Van der Pol oscillator, which consists of a closed curve (the thick line) that attracts the neighbouring trajectories.

In 1881, four years before the competition was organised, Poincaré was already thinking about applying this new qualitative theory to the study of the three-body problem and the problem of planetary stability. It is not in vain that the leitmotifs of the article *Memoir on Curves Defined by Differential Equations* were Poincaré's questions: "Does a closed curve describe a point that moves? Does it always remain inside a certain portion of the plane? In other words, and in the parlance of astronomy, we are asking if its orbit is stable or unstable."

Years earlier, in 1878, the attention of the American astronomer George W. Hill had also been attracted by the importance of finding periodic solutions (closed curves) in relation to stability. In fact, a periodic movement (repeated over and over again) provides us with a highly useful test of stability, since we know that the moving body can never escape from its orbit and collide with another object or escape to infinity. In fact, Hill had discovered a periodic solution to the three-body problem when the mass of one of the bodies was negligible with respect to the other two.

Studying Hill's problem, which is a 'restricted' version of the three-body problem, in which a light planet moves under the gravitational attraction of two equal stars confined to the same plane, Poincaré showed that, just like the general three-body problem, it cannot be solved using the classic methods for solving differential equations (quadratures). This is because, in contrast to the two-body problem (solved in this way by Newton, Bernoulli and Euler), not all the integrals of motion can be solved with the help of the laws of conservation (energy, momentum...). Thus Poincaré deduced that there is no explicit general solution using simple and known functions.

Poincaré was left with just one option – the perturbation method. When he applied it, he obtained solutions in the form of infinite power series. Yet it was far from clear that these series (like the corresponding series obtained by Euler, Lagrange and Lindstedt) were convergent, although they satisfied the equations of the three-body problem. In short, analysis had abandoned him to their fate.

It was not until 1909, some 20 years later, that the mathematician Karl F. Sundman (1873–1949) was finally able to give a general solution to the problem using a convergent series. Yet the solution is just as complicated and converges so slowly that in practice it is completely useless. However, in spite of this, if Sundman had arrived at this result 20 years earlier, he would probably have won the prize offered by King Oscar II.

Hence, abandoned by analysis, Poincaré sought the help of topology: "Tackling the problem from another point of view, I shall provide a rigorous proof of

the existence of periodic solutions." And since the stability of the solutions cannot be seen by examining the series, he returned to his qualitative theory of differential equations. Do these solutions determine closed curves, or in other words periodic solutions? If a moving body travels the path of a closed curve, or rather, a cycle, sooner or later it will need to repeat the same motion, which will hence be periodic.

Equipped with this new tool, to which he himself gave birth by fusing analysis and topology, Poincaré proved the existence of infinite closed curves and hence infinite periodic solutions.

King Oscar II of Sweden and Norway (left); Gösta Mittag-Leffler (right).
"A Pythagorean king and a Platonic mathematician."

And the winner is...

Twelve mathematicians entered the competition organised by King Oscar II and of the 12 submissions, only five tackled the three-body problem, – although none discovered the solution requested in the form of a power series. Hence, on 20 January 1889, the day before the monarch's birthday, the distinguished panel of judges, with the king's approval, declared Poincaré the winner for his *Memoir on the Three-Body Problem and the Equations of Dynamics,* which said:

> "This memoir cannot be regarded as a solution to the problem proposed, but it is of such importance that its publication will usher in a new era in the history of celestial mechanics."

The result was immediately published in the international press, and Poincaré was regarded as little less than a hero by the French press, which saw his victory as a triumph of French mathematics over German mathematics, which was traditionally considered dominant.

However, the signs that all was not well soon became clear. When Mittag-Leffler, who had encountered more than considerable difficulties when it came to reading Poincaré's long memoir, made its content public, the astronomer Hugo Gyldén did not take long, together with Kronecker, to claim that the work by the French mathematician did not go any further than his own, published in 1887.

However, the situation became even more complicated when, months later, in July 1889, Poincaré was struck by a barrage of questions from Edvard Phragmén, the editor of *Acta Mathematica*, seeking to illuminate the more opaque passages of the extensive 'memoir' with a view to its publication. Not without reason, Hermite wrote: "In this work, like in all his research, Poincaré shows the way and gives the directions, but leaves much to be desired when it comes to filling in the gaps and completing his work."

Furthermore, at the end of November, the author himself discovered a fatal flaw in part of his work, which he communicated to Mittag-Leffler in a letter dated 1 December:

> "I wrote to Phragmén this morning to advise him of a mistake I made and I doubt he will show you my letter. However, the consequences of this error are more serious than they might at first seem. It is not certain that doubly asymptotic solutions [the separatrices of the saddle points] are closed curves... and, hence, periodic solutions. What is certain, however, is that the two components of this curve [each of the two separatrices] intersect each other a finite number of times. I shall spare you the details of the irritation that this discovery has caused me. A considerable number of changes are required."

The letter was, of course, the worst type of letter that can be received by the editor of a journal and the organiser of a prize, since it represents a serious loss of

prestige for the panel of judges and the organisers. Mittag-Leffler's position became nothing short of untenable. He attempted to recall each and every printed copy of the memoir that was in circulation without publicising Poincaré's error, so that when it did become public, the damage caused to him by the scandal would be limited. Hence, an entire edition, printed but not published, of the prestigious journal *Acta Mathematica* had to be destroyed. (In fact, just one example of the original issue of the journal has been preserved, carefully guarded in a safe at the Mittag-Leffler Institute.) Meanwhile, in just two months, December 1889 and January 1890, Poincaré hurriedly reviewed and corrected his work in full, sent it to the printers and paid for it out of his own pocket, since he had agreed to accept responsibility for the expense caused. It is estimated that he paid more than 3,500 Swedish kroner, considerably more than the 2,500 kroner he would have received in prize money. A fine example of intellectual honesty.

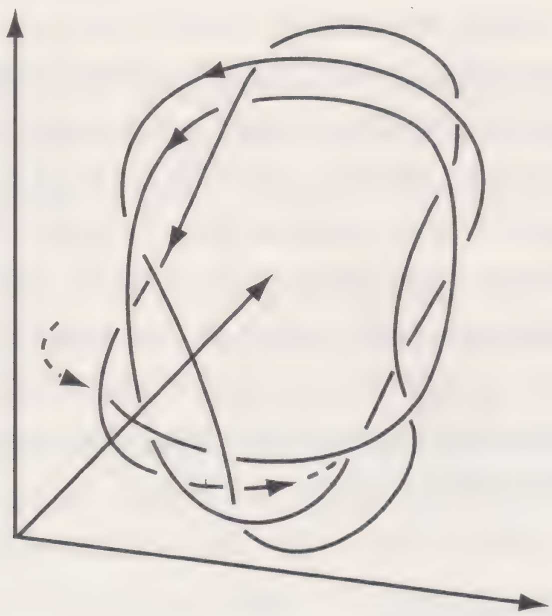
A monster called Poincaré

What exactly was Poincaré's error? The French mathematician had announced the discovery of an infinite number of periodic solutions for the three-body problem but, as he later acknowledged in his letter, some of these were not periodic as they did not define closed curves. In fact, based on this serious error, Poincaré was able to discover that the doubly asymptotic trajectories, the separatrices of the saddle points (or homoclinic points as he named them), determined chaotic orbits.

Let us consider this in greater detail. On the two-dimensional plane, Poincaré and Bendixson were able to prove their reassuring theorem due to the plane's especially favourable properties. Since the trajectories on the phase plane cannot intersect, there is an extremely limited number of valid behaviours. In fact, as we have already seen, there are essentially five: move towards or away from a point; be a sink, source or saddle point; or periodically orbit around a centre or limit cycle.

On the other hand, for the case of three bodies moving under mutual gravitational interaction (the three-body problem), we find ourselves with a problem in space, in three dimensions, which allows a greater number of combinations. Things are even more complex in the phase space. The trajectories do not need to intersect in order to cause headaches because they can also form knots without ever meeting. There are no knots on the plane, although they are present in space. Furthermore, when there are more than two dimensions, the system can exhibit attractors that are extremely

different to the singular attractive points (sinks) and limit cycles. As we shall see in the next chapter, 'strange attractors' commonly associated with chaos, appear.

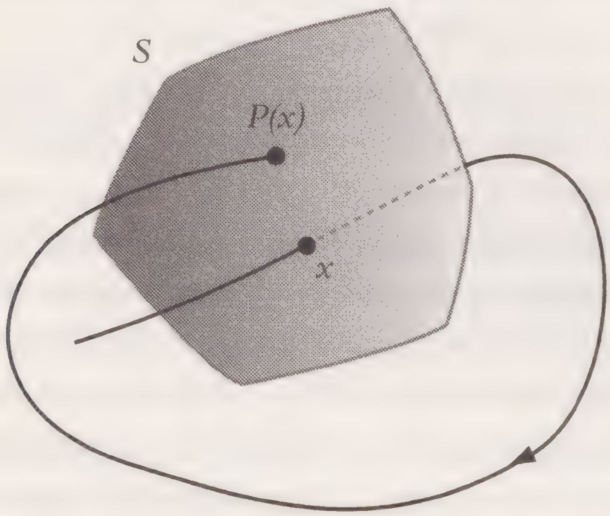


In three-dimensional space, knots can form between the trajectories of solutions.

How did Poincaré manage to detect periodic solutions in space? He did so by using the method now known as 'Poincaré sections'. Since it is easier to study the dynamic on the plane than in space, he decided to consider a plane that lay in the phase space and completely cut through the three-dimensional tangle of the trajectories. In other words, something extremely similar to what we do in everyday life if we want to know if there is a worm inside an apple: we cut the apple with a knife and look at the section. However, we shall explain things using a less unpleasant example.

Imagine that a person carries a reel of thread for a whole day, from sunrise to sunset, leaving the thread wherever they go. Of course, this thread is no more than an indication of the trajectory that the person has followed. However, let us now imagine that due to unforeseen circumstances we lose track of the person and do not know if they have returned home. How can we check if they are at home without needing to follow the thread all the way back to the reel and the person. At this point, Poincaré's topological approach comes to our aid. The plane on which the door of their house lies will be our Poincaré section. We stand in front of the door and count the number of threads that cross the threshold. If the number is odd, the person is still not home. However, if the number is even, we are in luck – they are

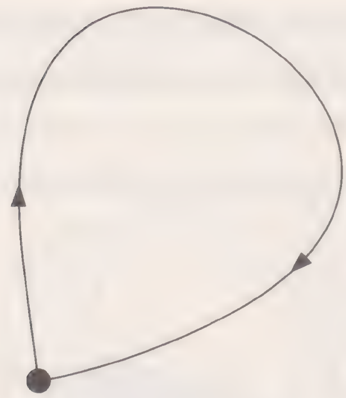
inside. (Logically, if they are in the house, they must have gone out and come back and hence, there must be an even number of threads crossing the threshold of the door, our Poincaré section.) To summarise, we can obtain important information from studying the threads (the trajectories) that cross a surface, such as the frame of a door (the Poincaré section).



Poincaré section S. If x and P(x) have coincided, the trajectory will form a closed curve and hence a periodic solution.

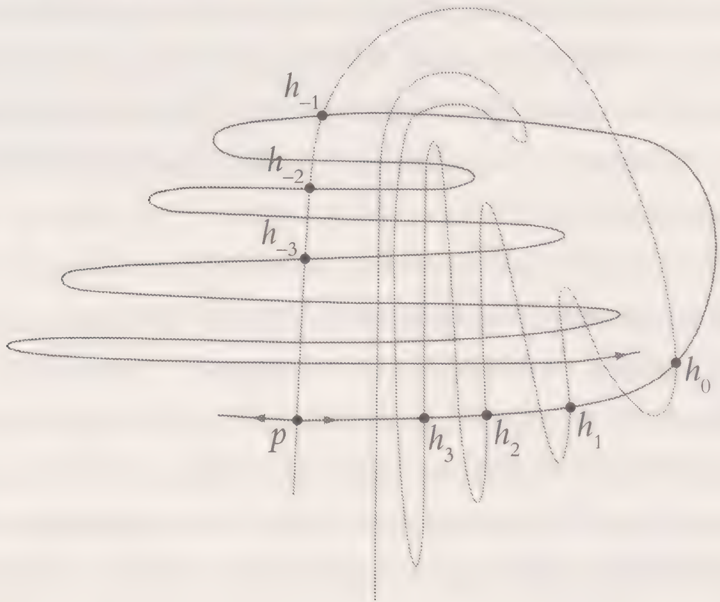
In general, Poincaré used the fact that the periodicity of a solution can be detected by using a Poincaré section, checking that the curve returns to exactly the starting point at which it crosses the section. Hence, the Poincaré section of the phase space captures crucial aspects of the solutions to the differential equation (including notions of stability).

Essentially, the French mathematician believed the dynamic drawn on each section would be a typical two-dimensional dynamic that was not too complicated, in which the trajectories could only cross at singular points (the saddle points). However, he discovered to his horror that in spite of the fact that the separatrices of the saddle points intersect (the two trajectories that collided on the homoclinic point), they do not coincide, but are two different curves that are obliged to cross each other repeatedly, forming a grid made up of an infinite number of points of intersection. The three-dimensional dynamic projected on each section exhibited a complexity that was unimaginable – at least to begin with.



Poincaré's error lay in believing that the unstable separatrix (which moves away from the saddle point) and the stable separatrix (which moves towards it) coincide (are the same curve).

Hence, the key lies in the fact that while the local structure of a saddle point is simple due to its linearity, there is no reason why this must be the case for the overall structure, since it is non-linear. Furthermore, it may be extremely complex, and herein lies the reason for the chaotic motion. The overall structure of the separatrices, or doubly symptomatic solutions (as they were referred to by Poincaré, since they moved from one singular point to another) can be, as we have seen, extremely complicated. In the case of the three-body problem, both separatrices are forced to cross each other over and over again, indefinitely. This is the 'homoclinic tangle', Poincaré's great discovery, a shape of such complexity that the author himself dared not draw or describe it. In fact, it is the root of chaos and the fact that the system lacks analytic integrals.

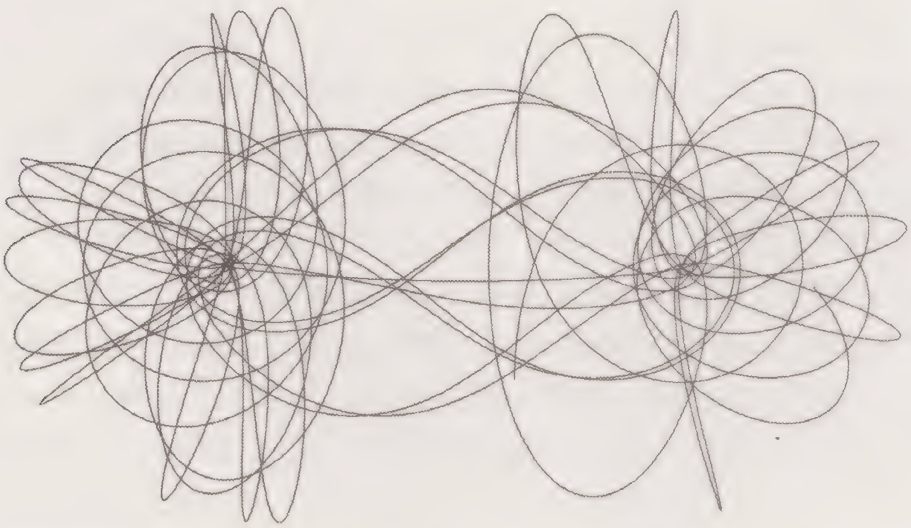


Homoclinic tangle: p is the saddle point, and h_0, h_1, h_2... are the infinite homoclinic crossing points between the two separatrices.

Subsequently, in his monumental essay, *Les méthodes nouvelles de la mécanique céleste* (New Methods of Celestial Mechanics), published in three volumes between 1892 and 1899, Poincaré provided the first mathematical description of the chaotic behaviour of a dynamic system with respect to homoclinic orbits:

> "When one attempts to draw the shape made up by these curves and their infinite number of intersections, each of which corresponds to a doubly asymptotic solution, the intersections form a sort of network, a spider's cobweb, or infinitely fine and tangled grid. Neither of the two curves is even able to cut itself, but must fold over itself in a highly complex manner to be able to cross the links of the tangle an infinite number of times. One feels a sense of vertigo when faced with the complexity of this shape, which I do not even dare to draw. There is nothing that can give us a better idea of the complexity of the three-body problem."

The homoclinic tangles are the hallmark of chaos, and the more than 200 pages of Poincaré's prized and corrected memoir constitute the first manual or textbook on chaos theory. "He seems," writes Hermite, in a letter to Mittag-Leffler, "like a clairvoyant to whom truths appear with an intense light – but only to him."



Chaotic orbit in the restricted three-body problem. If instead of living in orbit around a single sun, we lived on a small planet that moved around a double star, Kepler would have found himself obliged to abandon his aim of finding regular laws for planetary motion, since the planets rotate around each star according to periods with no regular pattern.

In this respect, Poincaré contributed, like few others, to the idea that there are deterministic dynamic systems whose prediction is impossible for researchers. The solution-trajectories of a differential equation can be so tangled up with each other that the slightest error when it comes to selecting the correct trajectory, which represents the solution to our problem, can cause us to pick the wrong trajectory and follow another that leads us to a completely different place or final state. In his *Science and Method* (1908), while using the three-body problem as a basis and, somewhat curiously, weather forecasting, Poincaré concluded:

> "If we knew the laws of nature and the situation of the Universe at the initial moment exactly, we would be able to provide an exact prediction of the situation of the Universe at a subsequent moment. However, even when these natural laws do not hold greater secrets for us, it is impossible to be more than 'approximately' aware of the initial situation. If this allows us to predict the subsequent situation 'with the same approximation', which is all that we need, we say that the phenomenon has been predicted and that it is governed by laws. However, this is not always the case. It may happen that small differences in the initial conditions give rise to extremely large ones in the final outcome. A small error in the beginning will give rise to an extremely large error in the end. Prediction becomes impossible."

Furthermore, a few months before his death, in 1911, upon his return from the Solvay Conference, where he learnt about quantum theory from Max Planck (which, together with chaos theory, constitute the two great cracks in the concepts of determinism and predictability), Poincaré was so concerned as to write:

> "It would seem unnecessary to point out that these ideas differ from traditional concepts; physical phenomena will stop conforming to laws that can be expressed using differential equations and this, undoubtedly, will be the greatest and most radical revolution in natural philosophy since the times of Newton."

Upon asking whether differential equations are a suitable instrument for mathematically formulating the laws of physics, the brilliant Poincaré was expressing nothing short of his doubts regarding the validity of determinism in the manner of a mathematician.

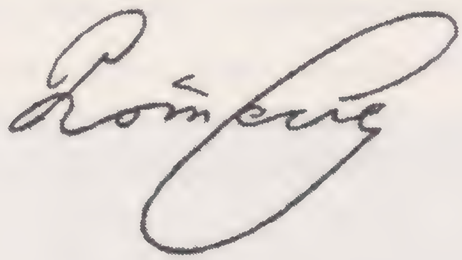
Newton had dressed determinism or the law of causality in mathematical clothing: Newton's laws were differential equations. Thanks to the development of a series of techniques for calculus by mathematicians, classical mechanics had achieved a great predictive power. However, Poincaré had now shown that certain mechanical systems could exhibit a motion that was so complex and chaotic as to make prediction impossible. And to complicate things further, not only did the scientific capacity for prediction have limitations, but quantum physicists themselves were questioning the mathematical clothing of determinism: differential equations. With the passing of the 20th century, both revolutions (chaos theory and quantum mechanics) would be firmly established.

JAMES CLERK MAXWELL: BETWEEN CHAOS AND ELECTROMAGNETISM

Influenced by the observations of the French engineers Saint-Venant and Boussinesq, on 11 February 1873, the famous Scottish physicist James Clerk Maxwell (1831–1879) gave a lecture on determinism in Cambridge that showed the extent to which he was aware of the footprint of chaos – just like Poincaré – a concept now known as the 'butterfly effect' or 'sensitive dependency on initial conditions':

> "Much light may be thrown on some of these questions by consideration of stability and instability. When the state of things is such that an infinitely small variation of the present state will alter only by an infinitely small quantity the state at some future time, the condition of the system, whether it is at rest or in motion, is said to be stable; but when an infinitely small variation in the present state may bring about a finite difference in the state of the system in a finite time, the condition of the system is said to be unstable. It is manifest that the existence of unstable conditions renders impossible the prediction of future events, if our knowledge of the present state is only approximate and not accurate.
> "If, therefore, those cultivators of physical science from whom the intelligent public deduce their conception of the physicist, and whose style is recognised marking with a scientific stamp the doctrines they promulgate, are led in pursuit of the arcana of science to the study of the singularities and instabilities, rather than the continuities and stabilities of things, the promotion of natural knowledge may tend to remove that prejudice in favour of determinism, which seems to arise from assuming that the physical science of the future is a mere magnified image of that of the past."

Today, a century later, the progress made by Poincaré with respect to his contemporaries is still amazing. Never has a mathematical error, in spite of the fact that it came in the midst of a competition, been so fortunate or fruitful. Thanks to all these contributions, he is frequently regarded as the grandfather of chaos theory. If Poincaré laid the foundations, Smale and Lorenz, much later on, finished the construction. This is why, together with other figures, they are regarded as the founders of chaos theory. However, we are getting ahead of ourselves…



Signature of Henri Poincaré.

Chapter 2

The History of the Rediscovery of Chaos

> *"You are not a normal case."*
> *"No?"*
> *"No."*
> *"What am I then?"*
> *"A case study."*
> *"I may be a case study; but nobody wants to study me."*
>
> *Pío Baroja,* The Tree of Science

Nobody knows or has known anything new immediately. What we think we know has suddenly, in fact, been with us for a long time. – perhaps, in the case of chaos, in an almost clandestine manner, without coming into the light since no scientist has wished to tackle something that does not look promising head on. This brings to mind an anecdote involving an American physicist that provides an excellent summary of why the route to chaos discovered by Poincaré went practically unexplored for almost half a century – meaning it needed to be rediscovered.

Doyne Farmer, a physicist and mathematician who achieved fame in the United States for beating the roulette wheels in the casinos of Las Vegas thanks to a non-linear differential equation, recounts his experience as a mathematics student:

> "*Non-linear* was a word that you came across at the end of the book. A physics student followed a mathematical curriculum and the last lesson was on non-linear equations. It was common to skip over the topic, and if it was taught at all, the only thing that was taught was how to reduce non-linear equations to linear equations to find approximate solutions. It was a frustrating exercise. We had no idea of the great differences that non-linearity produced in a model. We did not know that a non-linear equation

could evolve in an apparently random manner. In the event of observing a similar phenomenon, we asked, 'where is this random motion coming from? I can't see it in the equations."

However, between Poincaré and the new theories of chaos we shall discuss on the following pages, there were other exceptional mathematicians and physicists who, at the time (the last years of the 19th century and the start of the 20th) also studied the Frenchman's work on the productive *ménage à trois* that was the three-body problem. These explorers of chaos followed the call to refute Poincaré when it came to tackling non-linear problems, giving rise to new discoveries in similar fields. One of these explorers was Jacques Hadamard.

Although various examples of chaotic systems had been gathering momentum for a while, it was Hadamard who, in 1898, offered the first mathematical

LIVING LAS VEGAS

At the end of the 1970s, two physics graduates, Doyne Farmer and Norman Packard, founded a small group called Eudaemonic Enterprise, whose goal was to find a way of winning at roulette to raise money to fund scientific research. By studying a game of roulette they had purchased, they devised an equation that included the periods of rotation for the roulette wheel and the ball that travelled around it. As the calculations for solving the equation were extremely laborious, they decided to build a small computer to predict the octant (eighth of a circle) in which the ball would come to a rest. The computer was small enough to be hidden in the sole of a shoe, and the user was advised of the prediction on which they should bet thanks to three vibrating solenoids hidden under their clothing and attached to their body.

In 1978, the group set off for Las Vegas with the intention of winning serious money. As the observer supplied the information to the computer, the woman making the bet received the prediction under her clothes. They achieved an average return of 44% on their stake. However, the adventure was not without problems. One of the nights they played, the electrical insulation failed and the person making the bets received electric discharges from the solenoids that burnt their skin, although they continued to play with a straight face. In the end, the group won around $10,000. They had achieved their goal: to be able to statistically predict the area of the roulette wheel on which the ball would land.

A note of caution. This was not an easy experience and nor can it be applied to all roulette wheels. Under ideal conditions, a game of roulette with a perfect wheel and a perfect

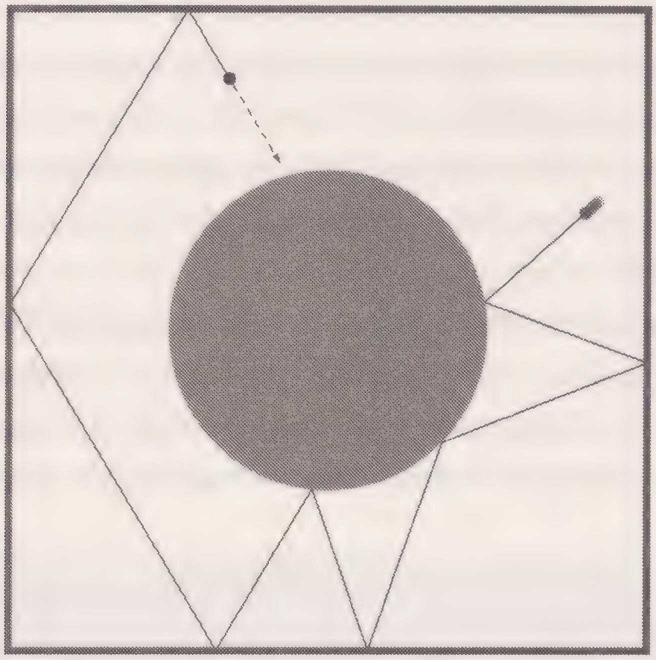
proof that, for certain dynamic systems, a small change in the initial condition could result in a large change in their subsequent evolution, and hence the final state (what we now call the butterfly effect). The French mathematician studied a sort of warped billiard table (curved, not flat, in the shape of a saddle) in which the trajectories of the balls were highly unstable, such that two balls initially placed close together, would tend, after the impact that set them in motion, to diverge considerably (in fact, exponentially) over time. He proved a theorem for this system, and others of a similar nature, on their sensitivities to initial conditions.

Many years later, in the 1970s, the Soviet mathematician Yákov Sinái (b.1935) returned to Hadamard's research and, instead of using a warped billiard table as a base, had the idea of studying the motion of the balls on a flat, square billiard table on which he had placed a number of disc-shaped obstacles. The mathematician proved

circumference would behave mathematically, making prediction impossible. If the 'Eudaemons' were able to make statistical predictions about the area of the wheel in which the ball would come to rest, this was a consequence of studying the imperfections of a specific roulette wheel. In short runs, the prediction emerged from the randomness due to imperfections in the materials. The roulette wheel's imperfections allowed them to make predictions.

The shoe-computer used by the 'Eudaemons'.

that his game of billiards had the same property as Hadamard's, since the obstacles also resulted in the chaotic dispersal of the balls.



The erratic trajectory of one of Sinái's billiard balls.

Another landmark discovery came at the hands of an old classmate of Jacques Hadamard, with whom Poincaré even had a metaphysical dispute. We are referring to the French physicist, Pierre Duhem (1861–1916), whose intense devotion to Catholicism led him to give philosophical priority to religion over science, a decision that could only be rejected by the rationalist Poincaré. Duhem considered the important philosophical repercussions stemming from the results proven by both figures, catching a glimpse of the revolutionary ideas that lay hidden behind their work. In the section 'Example of Mathematical Deduction that Can Never Be Used' in his work *La théorie physique: son objet et sa structure* (*The Aim and Structure of Physical Theory*), published in 1906, Duhem noted that the long-term prediction of the path of Hadamard's billiard balls is completely gratuitous, pointless, because any small uncertainty in the measurement of the initial position and speed of the ball will give rise to a spurious prediction. The predicted trajectory will bear no relation to the real one. In Duhem's book, we read:

> "Hadamard's research provide us with a highly representative example of this type of deduction, which is always useless. The example proceeds from

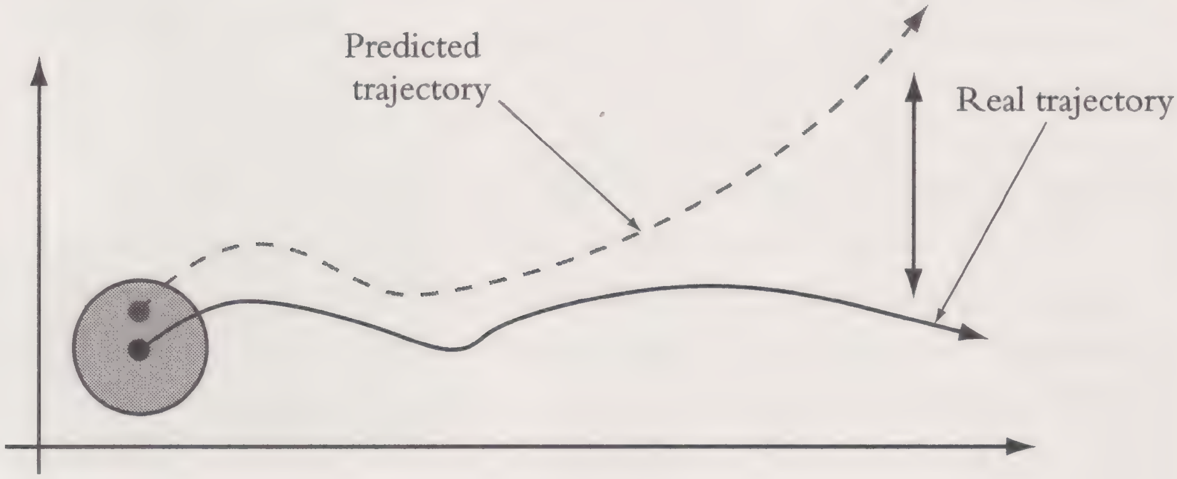
GRANDFATHER HADAMARD

Jacques Salomon Hadamard (1865–1963), a Gallic scientist of Jewish origin, who loathed mathematics as a child, held the Poincaré chair at the French Académie des Sciences when he died. Hadamard was the patriarch of Parisian mathematics, first as professor of an institute (it seems he was not very good and his students complained they could not understand him) and then as a university professor (although generally speaking he was the only person interested in the mathematical topics on which he worked).

A legendary example of his absent-mindedness is the fact that during World War II, with the Nazis ruling France, he left his emigration visa for the United States in his house in Paris when he travelled to America. Upon arrival, at the age of 79, he wanted to find a way to earn a living once again, and he went to a university where he was received by a professor who did not know his name. "Look, that's me there," he said pointing to his portrait in a series of photographs hanging on the wall. When he returned a week later to receive an answer, he was told that he would not be given a job and saw that his photograph had disappeared without trace. An adherent of communism, there are those who believe he was the author of the mathematical theorems subsequently published in the former Soviet Union and attributed to Karl Marx.

one of the simplest problems studied by the least complex physical theory – mechanics. A mass of material slides over a surface without being acted on by gravity or any other forces, and without friction to impede its motion. If our material point moves over a given surface, it describes a path referred to by geometers as a 'geodesic line' of the surface in question. Hadamard's research specifically considered the geodesics of surfaces with negative curvature. If we have a perfectly exact measurement of the initial position of a material point and the direction of the initial speed, the geodesic line that the point in motion will follow can be determined without any ambiguity. However, the situation is quite different if the initial conditions are not given mathematically but practically. The initial position of our mathematical point will no longer

be a given point on a surface, but an arbitrary point taken from inside a small mark; the direction of the initial speed will no longer be a straight line defined without ambiguity, but any of the lines that make up a narrow beam. In spite of these strict limits, it is always possible to take this practical data in such a way that the real geodesic moves away from the predicted geodesic chosen beforehand. Regardless of by how much the precision with which the practical data is determined is increased – for example, making the area of the initial position of the material smaller or the beam that represents the initial direction of the speed narrower – the geodesic that remains at a finite distance will never be free from these unfaithful companions that diverge indefinitely. If the data is determined by physical procedures, regardless of how precise these may be, the question remains and will continue to remain without an answer."



If chaos is present, the real trajectory of the system and the predicted trajectory will diverge in the medium to long term.

Furthermore, just after this passage, Duhem goes on to discuss another problem which is clearly similar to Hadamard's: the three-body problem. Echoing Poincaré's research, Duhem noted that the tangle between stable and unstable trajectories might mean that it is not possible to establish beyond doubt whether the trajectory of the planets is stable or unstable. In his own words:

"The three-body problem represents a terrible enigma for geometers. If, at a given moment in time and with mathematical precision, the position and speed of each of the heavenly bodies that make up the Solar System is known, it is possible to state that each body follows a perfectly defined tra-

jectory. Hence, the geometer can pose the following question: will these bodies continue to orbit the Sun indefinitely? Or, on the other hand, will a day come on which one of the bodies moves away from the group of its companions to become lost in the immensity? This question constitutes the problem of the stability of the Solar System, which Laplace believed he had solved, and whose extraordinary difficulty has brought modern geometers to prominence, especially Poincaré. However, it may be the case that the practical data the astronomer provides to the geometer is equivalent to an infinite number of theoretical values that are extremely close together but nonetheless different. Furthermore, among these values, there are some that keep all the bodies at a finite distance for eternity, while others launch some of these heavenly bodies into the immensity. If a circumstance like the one similar to that presented in the problem studied by Hadamard occurs, for the physicist, any mathematical deduction with respect to the stability of the Solar System is an unusable prediction."

However, in spite of the long shadow cast by Poincaré among French mathematicians, for a large part of the 20th century no serious attempts were made at serious research on the behaviour of orbits and homoclinic – or chaotic – tangles.

There are two good reasons for the surprisingly long interval that elapsed between the ideas of Poincaré and the modern study of the phenomenon of chaos. The first was the discovery of quantum mechanics, which shook the scientific community and focused the energies of various generations of physicists and mathematicians. If quantum mechanics caused chance to intervene in a new and intrinsic manner, why bother introducing it into classical mechanics via a sensitivity to initial conditions. However, chance was already there, it had just been forgotten. The second reason is that the ideas of Poincaré, Hadamard and Duhem were ahead of their time, since there was still no way of using them. Only with the advent of modern computers has it become possible to carry out the complex calculations and numerical analysis required by their results. Obviously, computers, which played a fundamental role in chaos theory, did not exist as we know them at the start of the 20th century.

Poincaré's successors in America

With the passing of the years, at the start of the 20th century Poincaré's work continued in two traditions: on one side of the ocean, the American tradition of

MAX BORN (1882–1970): THE STRUGGLE WITH CHAOS

Already a famous physicist – the father of quantum mechanics no less – Born returned to prominence thanks to the role played by the sensitive dependency on initial conditions in physics. Born questioned whether classical mechanics was in fact deterministic. In his answer, he disputed the model of highly unstable gas proposed by H.A. Lorentz in 1905 to explain the conductivity of metals. Essentially, each particle of Lorentz' gas behaved like a Hadamard-Sinái billiard ball, since as the particle (e.g. an electron) moved and collided with another set of obstacles (e.g. the atoms of the metallic body) its course suffered multiple changes whereby the slightest difference in the initial conditions resulted in two completely different subsequent states. Once more, with accurate information on the position and speed of the particle, it was possible to make accurate predictions of the state at other points in time (before or after). However, this is only the case when it is possible to take a perfectly accurate measurement of the position and speed.

In his acceptance speech for the Nobel Prize for physics in 1954, Born cited another highly illustrative example: imagine a particle that moves without friction in a straight line between two walls and whose impacts are perfectly elastic. It moves backwards and forwards at a constant speed equal to its initial speed, meaning that we can predict its exact position at a given moment in time given a precise measurement of its speed. However, if we allow for a slight imprecision in the measurement of the speed, the uncertainty in the prediction of the position at any subsequent point increases over time. If we wait a sufficient period of time, the imprecision will become equal to the total distance between the walls. For example, it is impossible to predict the position over a sufficiently long period of time, in the long term. The effect of the sensitive dependency on initial conditions is a type of classic determinism.

Birkhoff and Smale, and on the other side of the Iron Curtain, the Soviet tradition, headed by Kolmogorov and Arnold, successors of Lyapunov. Hence, the French mathematical influence did not disappear completely, although it would be a while before its ideas on homoclinic points would be continued, and they were lost for many years.

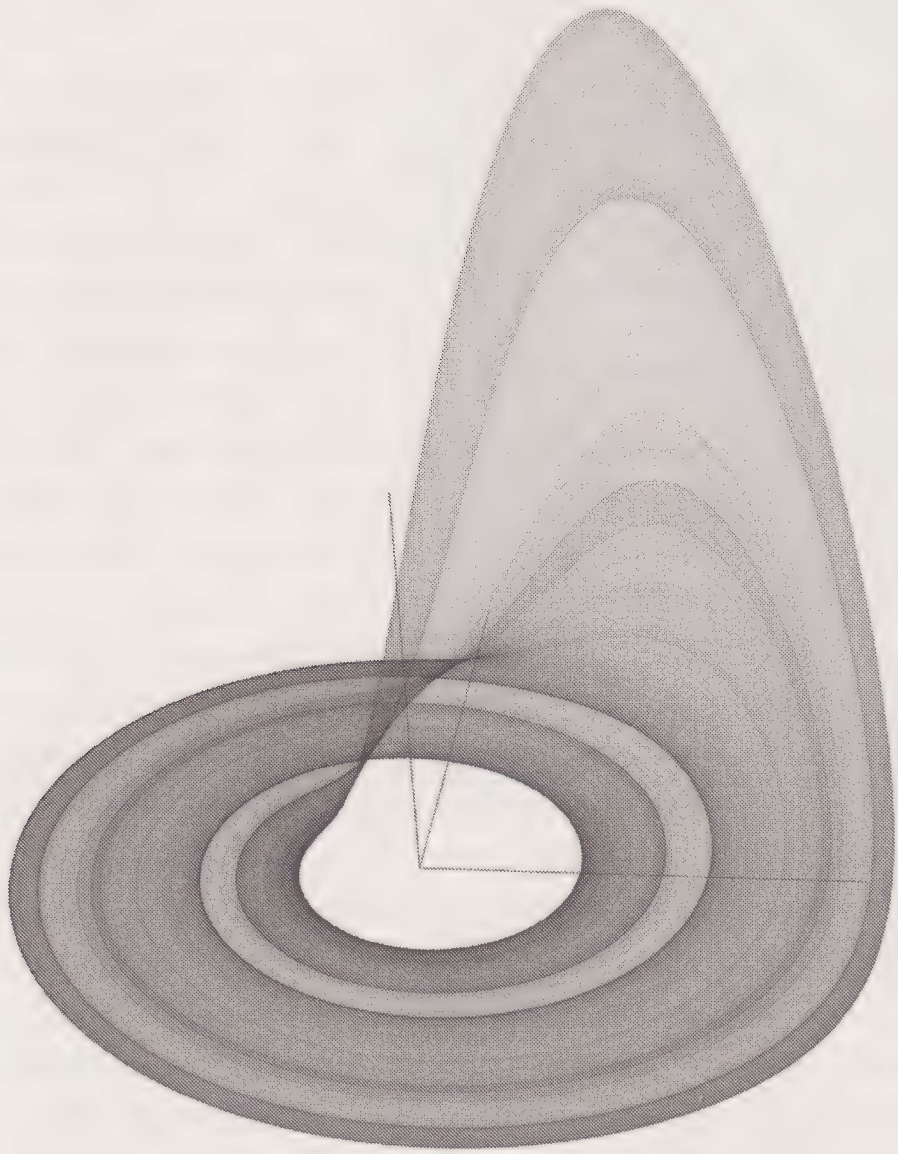
In the studies of George David Birkhoff (1884–1944), knowledge of Poincaré's work is clear in terms of the qualitative properties of differential equations. In his book *Dynamic Systems*, published in 1927, which introduces the term 'dynamic system' for the first time, this mathematician, who lived in the United States, developed the theory of these systems and went beyond the French mathematician in the analysis of curves defined by differential equations – in other words he took up where Poincaré left off and extended his ideas in new directions.

Also in the American arena, Stephen Smale (b.1930), winner of the Fields Medal in 1966, the highest accolade for mathematicians under the age of 40, stands out thanks to his great contribution to the theory of dynamic systems. Smale can be found at the confluence of the three major traditions in the study of these systems and chaos – the forgotten tradition dating back to Poincaré, passing through Birkhoff; the Russian school, which passed into English at the hand of Lefschetz during the Cold War; and thirdly, the analytic-topological study of differential equations carried out by Mary Lucy Cartwright (1900–1998) and John Edensor Littlewood (1885–1977) in Britain, based on the work of Van der Pol.

Balthasar van der Pol (1889–1959) was an electrical engineer from Holland who, in the 1920s, discovered a limit cycle – we have already come across this concept in Chapter 1 – for a non-linear differential equation describing the operation of vacuum tubes and electronic valves, and which plays an extremely important role in applications in telecommunications. There was a solution-trajectory to the equation in the shape of a closed curve that attracted all the neighbouring trajectories. In 1945, in the midst of the war effort, working together on mathematical aspects of radio and radar, Cartwright and Littlewood showed that the environment of this limit cycle displayed a complex, aperiodic movement... Once again, it was chaos!

Later, in the 1950s, the topologist Stephen Smale continued to study the qualitative behaviour of dynamic systems in search of a theory for three-dimensional space analogous to that of Poincaré–Bendixson's theory for the plane, but to no avail. And for good reason. It was, and still is, impossible, since trajectories in space can tie themselves in knots and this complicates the dynamic considerably. There are three-dimensional dynamic systems, which, aside from sinks, sources, centres,

saddles and limit cycles, exhibit strange attractors (to use an anachronism for the time). Unfortunately for Smale, chaos did exist.

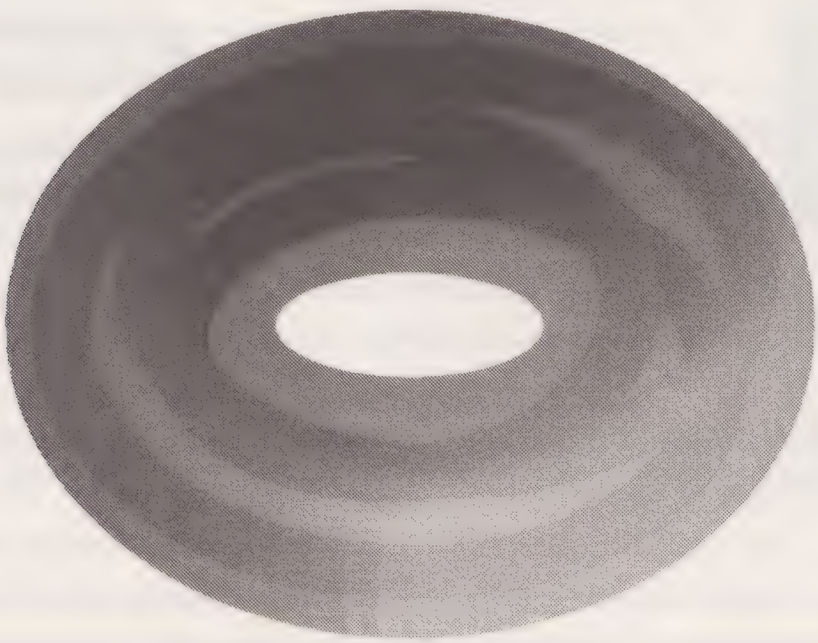


Rössler strange attractor (1976). Just like the Möbius strip, it has just one face, in spite of appearing to have two. Follow the outer edge to see that it changes into the inner edge.

At first, Smale believed that almost all (but not all) three-dimensional dynamic systems exhibited a behaviour that was not very strange, somewhat similar to the two-dimensional dynamic systems of the plane, whose only attractors (a highly descriptive term) were a finite collection of sinks and limit cycles. His interest in attractors lay in the fact that they determine the long-term behaviour of the dynamic

system, indicating what the system will do in the distant future, since they succumb to a sort of fatal or inevitable attraction to the attractor, as they approach infinity. Smale believed that the only movements valid over the long-term were those that represented a state of rest or balance in a stationary state (a sink), or repeated a series of movements periodically. In other words, staying still or going round and round; in short, points or circles.

However, imagine his surprise when, on the beaches of Río de Janeiro, he received a letter providing a counter-example to his conjecture. Norman Levinson, a fellow mathematician at the Massachusetts Institute of Technology (MIT), described a dynamic system like the one that gave rise to Van der Pol's non-linear oscillator, which had been studied by Cartwright and Littlewood, with an infinite number of periodic orbits and, worse still, a series of long-term behaviours in its neighbourhood that were most strange. In principle, it was possible that in the future, the system did not remain at rest or going round and round, but continued moving in a completely erratic manner. Providing a geometric representation of Levinson's analytic work, Smale first devised the 'Smale solenoid' in 1959 (so-called on account of its similarity to the apparatus in which miles of copper are coiled round a metal core to form an electromagnet) and secondly, in the 1960s, with the 'Smale horseshoe', whose highly complex dynamic is similar to the system proposed by Levinson. Two of the strangest attractors.

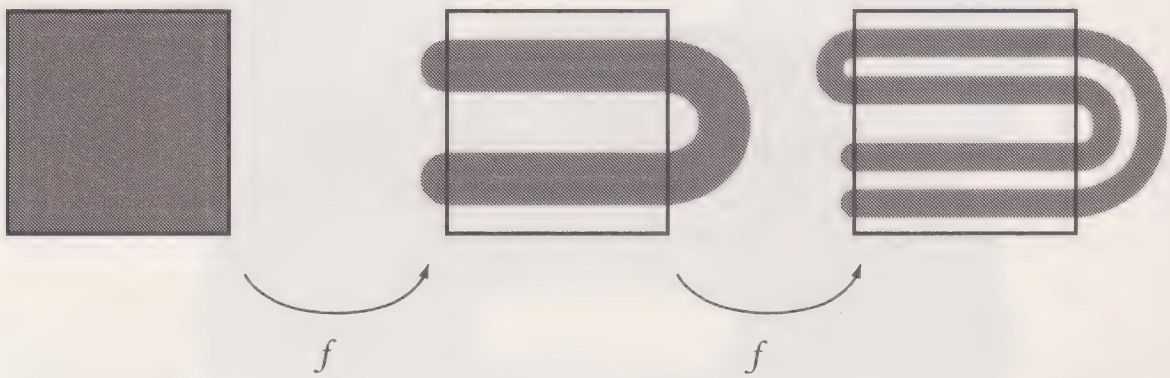


Smale solenoid, with a triple toroidal or doughnut-shaped coil within another, thanks to the fourth dimension.

The solenoid construction and, in particular, Smale's horseshoe represented an important step forward in understanding the relationship between a homoclinic orbit and the presence of aperiodic and unstable behaviour, which would later become known as deterministic chaos. Smale showed that the existence of homoclinic points implies the existence of a horseshoe, a shape that is paradigmatic of the topological mechanisms of stretching and folding which, as we shall explain in Chapter 3, give rise to chaos.

However, let us be bold and take a look. Both in the previous chapter and this chapter, we have approached chaos using intuitive geometric examples, knowing that it is often hard to understand what is going on. We know that it is common in popular science books, and even textbooks, to start with numeric examples and then go on to tackle geometric and topological illustrations.

In contrast, we have made a decisive choice to do just the opposite, and have done so for two reasons. Historically, this is how things happened, and because in this way, the reader can experience how mathematicians discovered what chaos was, step-by-step, first in a qualitative way and, much later on, quantitatively. If the reader feels their head reeling from such confusing descriptions, they should not worry, since they will only be feeling the same thing as the mathematicians of the time; with the advent of the computer, things became clearer.



Successive iterations of the Smale horseshoe. This consists of flattening a shape, stretching it and then folding it into a 'U' within the bounds of the original shape.

Both the solenoid and the horseshoe are, as we have mentioned above, examples of applications of geometric transformations that exhibit chaos. The transformation (let us call it f) that gives Smale's horseshoe is simple. To carry

it out, we begin with a square or a shape similar to a square. First, we flatten it a little and stretch it, before folding it into the shape of a horseshoe and fitting it into the boundaries of the original shape. Repeating this process *ad infinitum*, the transformation *f*, gives rise to a complex and intricate multi-layered structure, and chaos appears.

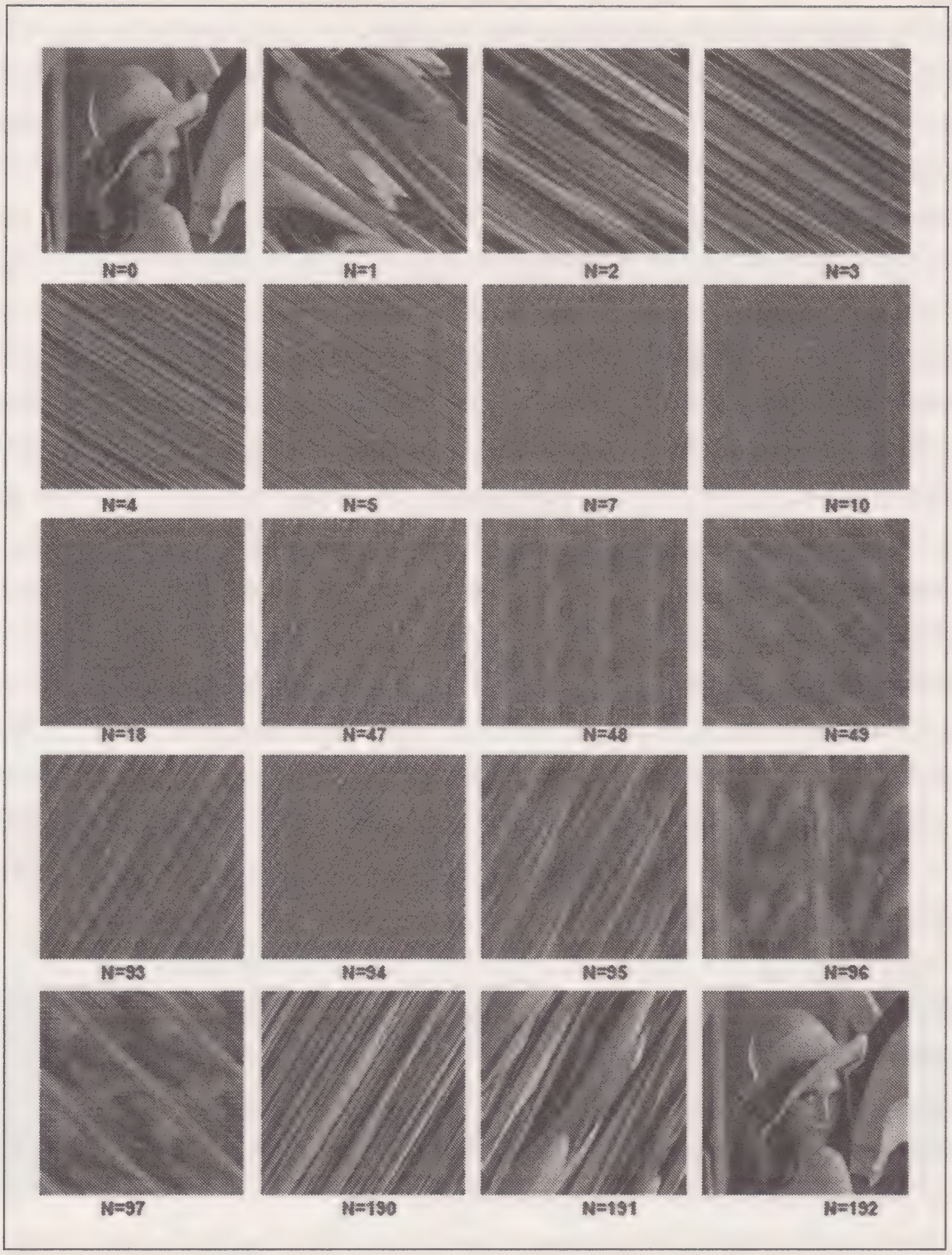
In the first iteration, the initial square is transformed into a sort of U-shaped horseshoe, as shown in the following figure. In the second, the horseshoe is turned into another horseshoe with three U-shaped curves. The third gives rise to seven curves with the same shape. And so on. At the limit, an infinitely convolved, folded curve appears, suggestive of the homoclinic tangle that terrified Poincaré. Effectively, stretching and folding is the essence of the geometry of chaos.

The successive stretches and folds that characterise the Smale horseshoe are a sign of chaos and, hence, appear in multiple chaotic maps. One example is the 'baker map', so-called because it reproduces the stretch and fold transformations made by bakers when kneading bread. Another is the 'cat map', named by its creator, V. I. Arnold (about whom we shall soon read more), which involves stretching and folding the face of a cat a number of times. However, we are not going to stretch and fold the face of a poor cat, not even Schrödinger's quantum cat, but are going to use a more attractive face instead – the face of Lenna.

Lena Söderberg was 'playmate' (Miss November) in the November 1972 issue of *Playboy* magazine. A section of one of her photographs (now mislabelled as Lenna) has been used since the 1970s as a test image for image compression algorithms, and has become the *de facto* industrial and scientific standard. (Who says mathematicians are nerdy!) In fact, the issue of *Playboy* in which Lena appears in the centrefold has become the best-selling ever.

Indeed, if we apply the cat map repeatedly to the Lenna image, or in other words if we repeatedly stretch and fold it in a specific way, we can see that the face completely disappears after a few iterations.

However, after a certain number of iterations (exactly 192), Lena's face reappears. Or more correctly, an extremely similar face appears (the trajectories of dynamic systems cannot coincide twice, except if they are periodic, and in this case we are dealing with a chaotic orbit). However, Lena's face reappears only to disappear again, and so on *ad infinitum*. Turn over to appreciate the wonders of chaos.



Application of the cat map to Lenna. The stretching and folding of the image (upper panels) ends up producing a homogeneous field (central panels), although some of the points occasionally return to being close to their original positions and cause a fleeting reappearance of the original image (bottom panels).

The worst thing (or best, depending on your point of view) that can happen to a dynamic system is that it can be chaotic. Under these circumstances, trajectories that are close together diverge rapidly from each other as they are stretched, compressed and folded in their approximation to the attractor. This determines highly strange

and complicated behaviour, as we have just seen in the case of Lena. This peculiar behaviour is the consequence of Poincaré's recurrence theorem.

In his work on the new methods for celestial mechanics, Poincaré had managed to formulate a surprising theorem: "Given the equations of a defined shape and a specific solution to these equations, it is always possible to find a periodic solution, whose period may in fact be extremely large, such that the difference between the two solutions is as small as required." The portrait of Lenna illustrates the discovery of Poincaré's recurrence. If a transformation is repeatedly applied to a system, and the system is unable to move beyond a limited region, it must return to a state close to the original state an infinite number of times; in other words, sooner or later, everything returns.

This is the mathematical version of Nietzsche's eternal return. The existence of a perpetual periodic solution implies that if the wheel of a bicycle suffers a puncture, we can simply wait for it to re-inflate on its own. If we wait for a sufficient period of time, the punctured bicycle tyre will refill with air. This is what Poincaré says. The only problem is that we might have to wait longer than the age of the Universe.

ARE YOU JOKING MR FEYNMAN?

Richard Phillips Feynman (1918–1988) was an eccentric US physicist who won the Nobel Prize for physics in 1965 for his contributions to quantum electrodynamics. He was a fan of hypnosis, topless bars and safe cracking. In his popular *Feynman Lectures on Physics* he left a note that leads us to ask the question: are you aware of chaos theory, Mr Feynman? In the section 'Philosophical Implications', in Chapter 38 of the first volume of his *Lectures*, published in 1965, Feynman described – just like Max Born – how, from a practical point of view, a sort of indeterminacy lies in the heart of classical mechanics as a result of the imprecision in specifying the initial conditions of certain physical systems. If we knew the position and speed of every particle in the world, it would be possible to predict what will happen in the future. However, let us imagine that we do not know the exact position of just one atom. This means that when this atom collides with another, the error in the position will have increased. And obviously after another collision, it will increase yet again, in such a way that the imprecision increases extremely rapidly. After a finite period, the imprecision will reach cosmic proportions.

Mathematics on the other side of the Iron Curtain

In parallel, crossing the iron curtain, we can find another fertile tradition – the Russian school, comprising many physicists and mathematicians who inherited Lyapunov's influential notions on the stability of motion in dynamic systems.

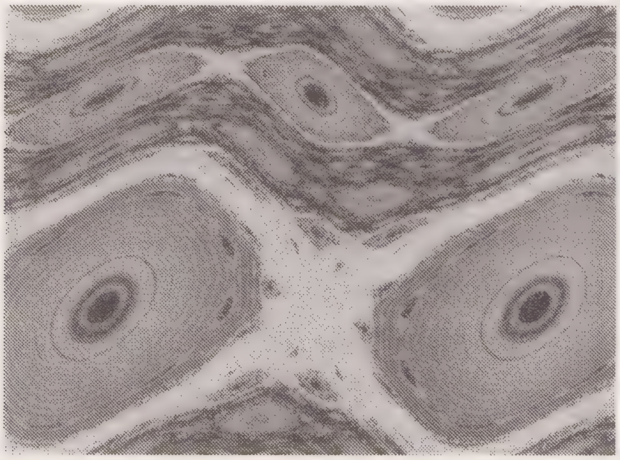
Working more or less during the same period as Poincaré, the mathematician and physicist Aleksandr Lyapunov (1857–1918) concerned himself with the theory of stability from a more quantitative perspective. Instead of studying the geometry of the trajectories, as the Frenchman had done, he studied certain numbers, called 'Lyapunov exponents', that can be used as indicators of instability. When one of these exponents was positive, the trajectories tended to separate (exponentially) with respect to each other. This meant that the system was unstable.

Under this precedent, in the 1950s Andrei Kolmogorov's (1903–1987) seminar at Moscow State University, focused on celestial mechanics, since both he and his student Vladimir Igorevich Arnold (1937–2010) focused on the theoretical study of the stability of celestial dynamic systems, taking up where Poincaré and Lyapunov had left off. As a result of this study, in 1954 Kolmogorov presented a theorem at the International Congress of Mathematicians held in Amsterdam: the K theorem.

Later, as the young German mathematician Jürgen Moser (1928–1999) wished to write a summary for the journal *Mathematical Reviews* and was extremely interested in the subject, he travelled to the Soviet Union and made contact with Kolmogorov's student, Arnold. The results of this joint research gave rise to the KAM (Kolmogorov–Arnold–Moser) theorem, which is famous among specialists. The theorem describes what happens when an integrable (or linear) system suffers a non-integrable (non-linear) perturbation and states that if the perturbations are small enough, the majority of the orbits behave in a stable and quasi-periodic manner, but they never diverge too much from the periodic orbits of the system. However, there are also others that are unpredictable, meaning the former form islands of stability in an ocean of chaos.

In the case of the Solar System, since the mass of the planets is insignificant in comparison to that of the Sun, in the first instance, it is possible to underestimate the forces between them, obtaining an integrable system in which each planet follows a beautiful Keplerian eclipse as shown by Newton. However, if we take into account the interactions between the planets, the problem is no longer integrable, as we know thanks to Poincaré; the motions of the planets no longer describe perfect ellipses and it is impossible for a planet to enter into a chaotic orbit that ejects it into outer space. Since 1954, and thanks to the KAM theorem, we know that small perturbations

only partially destroy the regularity, such that if we assume the interplanetary forces are not excessively intense, we can expect the majority of the planetary orbits to be regular, similar to Kepler's ellipses. However, this is not to say that all the movements within the Solar System must necessarily be regular, just that this must be true for the majority. Indeed, the destiny of certain minor members may well be to follow chaotic orbits that lead to a collision or to their escape from the Solar System. This will probably be the destiny of Chiron, a so-called centaur (i.e. half asteroid, half comet) that moves between Saturn and Uranus in an unstable eccentric orbit.



KAM theorem: islands of regularity in the midst of a sea of chaos.

Another illustration of the content of the KAM theorem is provided by the numeric study carried out by the French astronomer Michel Hénon (1931–) and the undergraduate student Carl Heiles (1939–) in 1962, with the help of a new tool – the computer. Both astronomers were studying the stellar dynamic, or rather, how stars move within a galaxy depending on their energy. At low energies, the solutions to the equations were, as was to be expected, periodic or quasi-periodic, whereas at high energies, the computers showed that the regular trajectories disappeared and were replaced by a sea of chaos in which an island of stability floated but could only be found floating every now and then. This was the Hénon-Heiles chaotic system.

However, the influences of the Russian school did not stop there. During the Cold War, the main discoveries of the Soviet mathematicians were translated into English and made available to other mathematicians, both European and North American, thanks to the providential work of Solomon Lefschetz (1884–1972).

The chemical engineer was born in Moscow, studied in Paris and settled in the United States to work, where, as a result of an accident in which he lost both hands (a chemistry experiment went wrong and exploded), he became interested in mathematics. Thanks to his studies, he managed to overcome the deep depression into which he had fallen after the accident. With the passing of time, he was even appointed to a post at Princeton, where he gave classes thanks to the help of two artificial, plastic hands, although every day he needed a student to place the chalk in his right hand and then remove it after class. After the end of World War II, his contact with Russian mathematicians was vital to the development of the theory of dynamic systems and with it, the theory of chaos that was then beginning to germinate.

Lorenz: a coffee, a computer and a butterfly

Let us now return to the United States, where in the 1960s, 1963 to be precise, a young meteorologist at MIT by the name of Edward Lorenz (1917–2008), one of Birkhoff's old students at Harvard, devised a model made up of three standard differential equations to describe the motion of a fluid under the action of a thermal gradient. In fact, the problem is a simplification of the study of convection in the atmosphere, in other words, how the flow of hot and cold air behaves when subjected to a notable difference in temperature. The hot air rises and then cools as it reaches the upper layers of the atmosphere, causing it to fall again. For certain values of the model's constants, differential equations represented the beginning of a non-stationary, critical convection.

As he searched for numeric solutions with the help of a computer, the Royal McBee LGP-30 (the world's first personal computer), he discovered, after returning from a cup of coffee (although there are those who maintain it was tea) that it exhibited dramatic, unstable and chaotic behaviour. The computer was printing a list of extremely peculiar numbers without any pattern. Lorenz thought he had made an error in running the programme, and repeated it a number of times. However, he always obtained the same extremely peculiar numeric simulations, lists of numbers that started off almost the same but ended up being completely different. By chance Lorenz, had stumbled upon the phenomenon of sensitivity to initial conditions, something that rendered his system, in practice, unpredictable. He noticed that the system was highly unstable with respect to the tiniest modification. A slight change in the initial conditions resulted in completely different final states,

since two extremely similar initial states could evolve in radically different ways. In his own words:

> "Two states that were imperceptibly different could evolve to two considerably different states. Any error in the observation of the present state – and in a real system, this appears to be inevitable – may render an acceptable prediction of the state in the distant future impossible."

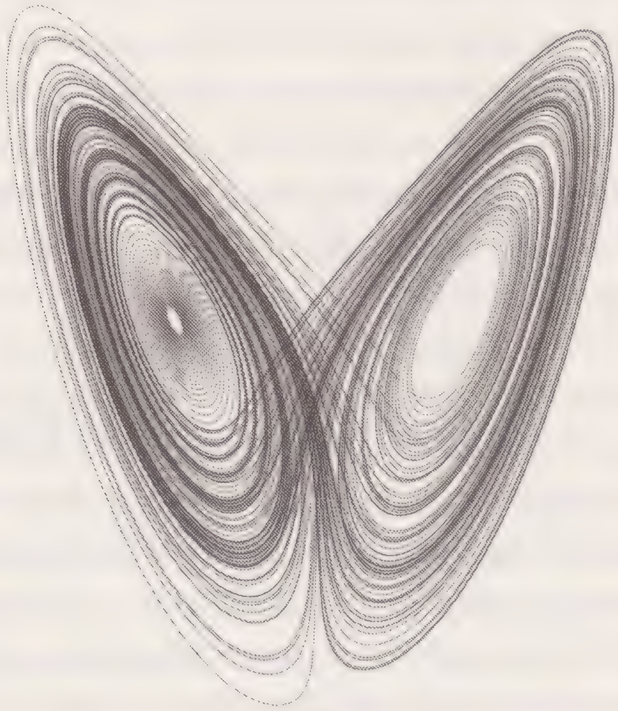
Borrowing the image that he would subsequently forge, Lorenz had discovered the butterfly effect: the flap of a butterfly's wings in Brazil could set off a tornado in Texas. Imagine that a small butterfly is resting on a tree in a remote region of the Amazon. As it stays at rest, it opens and closes its wings a couple of times. It could have done so just once, but in this case it has flapped its wings exactly twice. Since the atmospheric system is a chaotic system that exhibits a sensitive dependency on initial conditions, the slightest variation in the currents of air around the butterfly may end up influencing whether there is a hurricane over Texas some months later.

This phenomenon became famous in 1972, when, at the American Association for the Advancement of Science, Lorenz presented a paper entitled "Can the Flap of a Butterfly's Wings in Brazil Set Off a Tornado Over Texas?" However, when he came to report his research in 1963, he cited the comment made by a meteorological colleague in this respect: "Edward, if your theory is correct, one flap of a seagull's wings could alter the course of history forever."

While it was Lorenz who introduced the popular metaphor of the butterfly effect, it was the US mathematician Guckenheimer who coined the term "sensitive dependence on initial conditions" in the 1970s. At any rate, the result is the same: the chaotic dynamic causes trajectories that initially coincide to separate and diverge.

Just like the lists of numbers, the graphs that Lorenz reproduced in his article exhibited a series of oscillations that grew increasingly larger and became chaotic. To begin with, the trajectory was periodic, however after a while it began to oscillate violently, flipping between above and below without a fixed rhythm or constant pattern. The trajectories went round and round, apparently at random, forming a sort of figure in the shape of a figure of eight or butterfly. Sometimes they rotated a number of times around one of the eyes or lobes of the figure, whereas on other occasions they rotated around the other lobe for a different number of times. Furthermore, neighbouring trajectories tended to separate over time, as they were stretched and folded around the strange figure. The stretches that separated

trajectories that were together, increased the errors in the predictions, whereas the folds, for their part, contributed to mixing and confusing trajectories, bringing them close to different trajectories. This was the Lorenz attractor.



In contrast to classical attractors (points, limit cycles), which are predictable, strange or chaotic attractors, such as the Lorenz attractor, reproduced here, correspond to unpredictable movements and acquire more complex geometric shapes.

Lorenz published his findings in a meteorology journal, in an article entitled *Deterministic Non-Periodic Flow*, which went practically unnoticed. In fact, although Lorenz was a meteorologist, he had wanted to be a mathematician but had been prevented by the outbreak of World War II. Since his mathematical discovery was not of great importance to other meteorologists, the article remained practically forgotten in libraries for almost a decade.

Only Professor James Yorke (1941–) at the University of Maryland was able to recognise the scientific and philosophical repercussions of Lorenz's research, since a cursory glance at the bibliography of Lorenz's 1963 paper revealed that it brought together Poincaré's topological studies of non-linear systems, Birkhoff's theory of dynamic systems and – wait for it – the Soviet mathematical tradition, as detailed

in the book *Qualitative Theory of Differential Equations* by Nemytskii and Stepanov, published in 1949 in Moscow and translated into English in 1960.

If the butterfly effect (the sensitive dependence on initial conditions) and what we will call the 'mixing effect' (the stretching and folding of the trajectories) were more or less hidden in Poincaré's homoclinic tangles, both signs of chaos came to light with the Lorenz attractor and the Smale horseshoe, respectively. Strictly speaking, homoclinic tangles already set Smale onto the track of the solenoid and the horseshoe, whose stretching and doubling action is indicative and characteristic of chaos. Chaos theory had been re-born.

The new fathers of chaos theory

If Edward Lorenz offered the scientific community the paradigm of the continuous chaotic dynamic system (the Lorenz system), in his article *Simple Mathematical Models with Very Complicated Dynamics*, published in the journal *Nature* in 1976, the population biologist Robert May (1936–) laid claim to the paradigm of the discrete chaotic dynamic system (in which time was not continuous but progressed step-by-step). It was based on the logistic map, an extremely simple function: $f(x) = kx(1-x)$. However, when the value of k is close to 4, it exhibits, as paradoxical as this may seem, a highly complex dynamic. In the next chapter, we shall use this map to present and explain the main concepts of chaos.

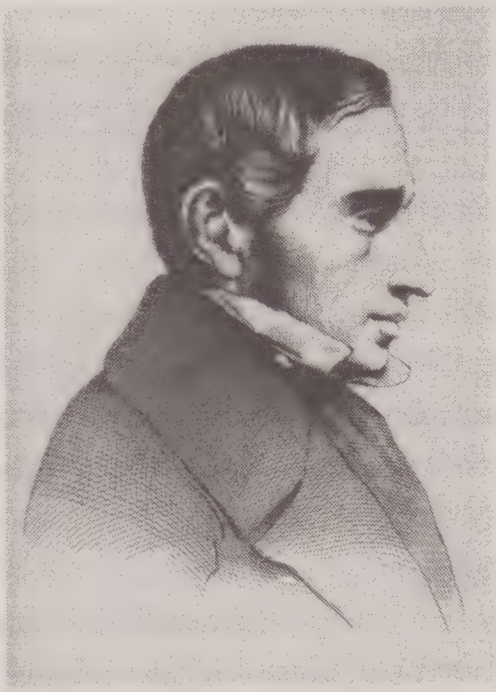
In fact, the term *chaos* was officially coined a year before May's publication. It was in 1975 when Professor Yorke introduced it into modern scientific literature thanks to his famous article *Period Three Implies Chaos* published jointly with Tien Yien Li. A few years later, between 1978 and 1979, the physician Mitchell Feigenbaum (1944–) heuristically discovered (research using less rigorous methods, such as trial and error) certain universal constants that characterised the transition from periodic motion to chaotic motion.

Finally, within this context, it should not be forgotten that between the end of the 1970s and the start of the 1980s, the exploration of the applications of chaos theory began to bear fruit beyond simulations on computer screens. A famous example of the importance of chaos in the study of physical phenomena is the transition to turbulence in fluids. Turbulence is an extremely important phenomenon, since it arises in many fields, from fluid mechanics to meteorology and climatology. For classical mathematicians, turbulence began as an accumulation of vibrations, and

the standard interpretation suggested that, as the motion of water in a river became faster, the sum of all the vibrations, despite being simple when considered separately, produced an unstable, changing and turbulent motion. The problem was that the majority of vibrations came to join forces, as they accumulated, giving rise to a combined periodic motion and not turbulence. Finally, in 1971, the mathematicians David Ruelle (1935–) and Floris Takens (1940–2010) sought to apply a different theoretical approach: to consider turbulence from the eyes of topology. It was then that a completely new and brilliant idea arose: the combination of vibrations can produce a new object, a 'strange attractor'. That receives its name on account of its strange geometry, since it is an attractor set, albeit an extremely different one to the classical examples known at the time (sinks and limit cycles).

Another application of chaos theory that is becoming increasingly important is related to the life sciences, where it has been successfully applied to the study of irregularities in heartbeats and the transmission of illnesses. Certain

AHEAD OF HIS TIME



Pierre-François Verhulst.

It is highly likely that the first dynamic system with which a newcomer to chaos theory will come into contact is the logistic map: $f(x) = 4x(1-x)$. In spite of appearing straightforward, the map exhibits a highly complex dynamic that includes chaotic behaviour. The logistic function is derived from the logistic equation first introduced by the Belgian scientist Pierre-François Verhulst (1804–1849). When, in a study on population growth published in 1838, Verhulst introduced the logistic equation to model the increase and subsequent slowdown of populations according to demographic statistics, he could not have imagined that more than a century later the model would attract significant attention as a paradigmatic example of chaos theory.

STRANGE ATTRACTORS VERSUS FRACTALS

The majority of strange attractors that appear in chaotic systems are fractal sets. To be more specific, fractal geometry, devised by Benoit Mandelbrot (1924–2010) in 1977 and based on the pioneering work of Fatou and Julia in 1918, has been regarded as the geometry of nature. Numerous natural structures (coastlines, the leaves of plants, shells, certain organs of the human body, such as lungs, galaxies, constellations… and even Saturn's rings, whose segments resemble the Cantor fractal set) conform to fractal designs. Self-similarity is an essential property of a large number of complex systems.

results in neuroscience and medicine offer even greater promise, such as the electroencephalogram, where the detection of chaotic and non-chaotic patterns (the latter being, curiously enough, abnormal) measured over a period of time appears today to be the only way of providing an early diagnosis of certain brain diseases, a topic to which we shall return in greater depth in the next chapter.

An overly noisy revolution

In spite of all that we have said until now, the impartial words (with a hint of scepticism) of David Ruelle in his book *Chance and Chaos* are still extremely pertinent:

> "The mathematical theory of dynamic systems has benefitted from the influence of chaotic ideas and, in general, has not suffered as a result of the evolution that is taking place… The physics of chaos, however, in spite of recent announcements of important innovative advances, has exhibited a decreasing production of interesting discoveries."

This makes no mention of the distorted vision of chaos that certain post-modernists and other esoteric thinkers tend to offer. Critical voices claim that disproportional coverage has been given to real chaos theory and fractal geometry. This abuse has led to their delirious application to literary analysis and business management.

CHAOS IN THE SKY AND ON THE LAND

If May introduced the paradigm of the one-dimensional chaotic discrete dynamic system (the logistic map), it was the French astronomer Michel Hénon who proposed the two-dimensional chaotic discrete dynamic system: the Hénon map. In 1976, years after Lorenz's work including the model of a continuous-time chaotic dynamic system had been published, Héron published an article entitled 'A Two-Dimensional Mapping with a Strange Attractor', in which he presented a transformation of the plane defined as:

$$H_{ab}(x,y)=(1+y-ax^2,\ bx),$$

where a and b are two constants that are often given the values $a=1.4$ and $b=0.3$. The map H represents a simplification of the Poincaré section of the Lorenz attractor. Indeed, if we repeatedly apply H to a square, we can see how it transforms, first of all into an increasingly flat quadrilateral as it is stretched and folded, and then into a sort of infinitely intricate horseshoe. This intricate structure (fractal), to which the iterations of H tend, is the Hénon strange attractor.

The structure of the Hénon attractor is a fractal, or in other words, self-similar (it is repeated on a small scale over and over again).

In spite of the fact that Hénon claimed it was a strange attractor (i.e. an attractor with a fractal nature), it was not until 1991 that the Swedish mathematicians Michael Benedicks and Lennart Carleson provided a rigorous proof of its existence.

What is undeniable, however, is that a new door has been opened – leading to chaos. This new multidisciplinary science is referred to as chaos theory or dynamic systems theory by mathematicians, non-linear dynamics by physicists and non-linear science by other scientists. The science of the butterfly effect, the sensitivity to initial conditions, random, erratic and irregular paths, aperiodic and unstable behaviour, homoclinic orbits, stretches and folds, strange attractors, and much much more. The time has come to cross this threshold.

Chapter 3

But, Mr Mathematician, What Exactly is Chaos Theory?

Yahweh: "Who can count Jacob's dust?
Who can count the cloud of Israel?"
Numbers, XIII 10
Mephistopheles: "Who knows how
the dice will land from now on?"

Johann Wolfgang von Goethe, *Faust*

God and the devil, for once agree: the human capacity for prediction has insurmountable limits… While Einstein's theory of relativity dispensed with the illusion of absolute space and time from Newton's classical physics, for its part, the quantum theory developed by Planck, Bohr and Heisenberg ruined the dream of controllable measurement processes. And chaos theory has demolished the fantasies of infinite predictability in one fell swoop.

However, the most significant break from traditional thought is the understanding that as a matter of principle it is impossible to predict the behaviour of many systems in the long term due to the extreme instability of the solutions to the equations governing their motion. It is a highly complex behaviour that is not caused by external noise or high levels of freedom, or even quantum effects. The equations are deterministic. However the solutions have stochastic properties. This is referred to as 'deterministic chaos'. In this chapter we shall attempt to provide a mathematical explanation of this concept. In the words of Charles Darwin "Mathematics seems to endow one with something like a new sense."

Chaos and complexity

For decades, chaotic and complex systems have been largely overlooked by institutionalised science. Twentieth-century science brought us close to the fabric of the Universe, to the spacetime of relativity and the microcosmos of quantum mechanics (the game's board), and contemporary science is now leading us towards a better understanding of how our reality is organised (the game's pieces). However, the true brilliance of science came to be valued for its usefulness and only recently, at the start of the 21st century, are we beginning to enter into the realm of chaos theory and the sciences of complexity.

In fact, chaos theory is just one of the so-called sciences of complexity: chaotic systems are just one type of complex system, and there are many more, such as fractal geometry, catastrophe theory and diffuse logic. The systems studied by chaos theory are regarded as being difficult to describe, standing midway between order and disorder, between crystal and smoke. While highly ordered systems (such as a crystal) or highly disordered systems (such as smoke), are simple and easily described, the systems in between represent a spike of complexity. Chaotic systems in particular are non-linear deterministic systems that exhibit aperiodic behaviour, making them unpredictable. There is a Chinese proverb that states that the flap of a butterfly's wings can be felt on the other side of the world, and Blaise Pascal wrote that if Cleopatra's nose had been slightly smaller, the history of the world would have been completely different: Octavia would have fallen in love with Cleopatra and would not have become the first emperor of Rome. Furthermore, as we shall see, chaotic systems are ubiquitous, appearing in mathematics, physics, astrophysics, meteorology, biology, medicine… Or rather, almost all, if not all, real systems have a chaotic dynamic.

Dynamic systems

We have already seen that chaos is a phenomena that is studied as part of the mathematical theory of dynamic systems. But what is a dynamic system? It is a mathematical model, generally used in the natural and social sciences, that consists of an equation describing the evolution of the state of a system over time.

There are discrete and continuous dynamic systems. In discrete systems, time varies step-by-step ($t=0, 1, 2, 3\ldots$). Hence, a discrete dynamic system is formally

defined by a difference equation, which is nothing more than a formula telling us how to calculate the next term from an initial term. This is then used to derive another term, and so on, giving an infinite series of numbers. To summarise, a difference equation is an equation of the form:

$$x_{n+1} = f(x_n),$$

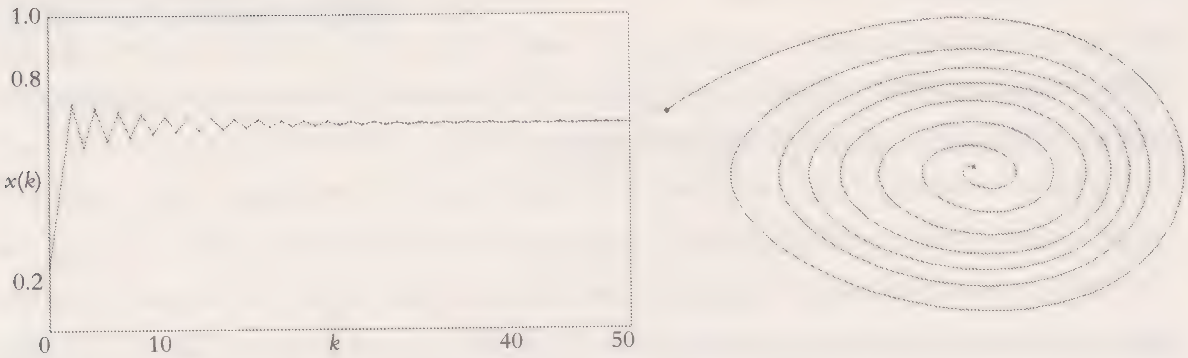
where f is a function telling us how to calculate x_{n+1} given x_n. In other words, how to calculate x_1 given x_0; x_2 given x_1; x_3 given x_2, etc. Hence, a difference equation is a formula expressing the value of the variable for the next step given its value for the previous step. Hence, given an initial condition x_0, the solution to the dynamic system is the trajectory or orbit $\{x_0, x_1, x_2, x_3, \ldots\}$ obtained by repeatedly applying f to x_0.

On the other hand, in continuous dynamic systems, time does not pass step-by-step, but does so continuously, just like in the real world. Continuous dynamic systems are described using differential equations, such as the ones we have seen in the previous two chapters, formulae that express the rate at which a representative variable changes in line with its current value. In the interests of brevity, our mathematical analysis of chaos will focus on the study of discrete dynamic systems, since these are the essence of the problem.

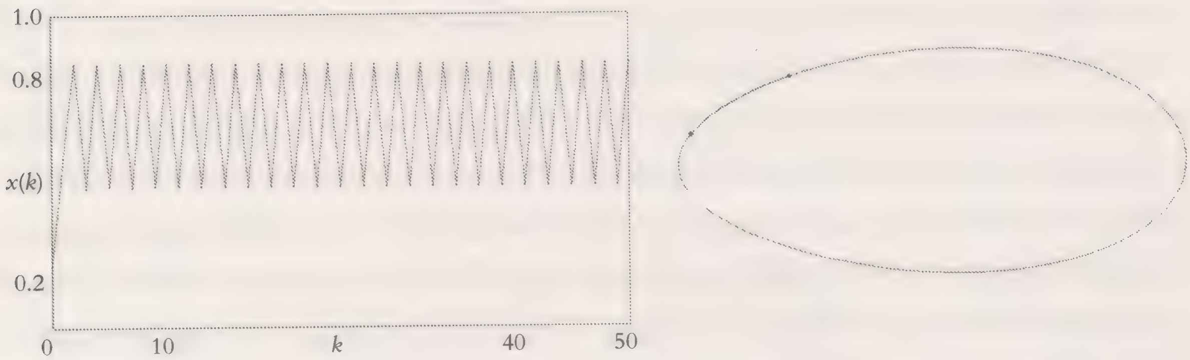
There is a mathematical theorem that states that a continuous dynamic system is chaotic if and only if there is a Poincaré cross-section for which it is possible to define a discrete dynamic system that is also chaotic.

Indeed, within discrete dynamic systems there is a class that exhibits an extremely important property – non-linear systems. A system is linear if the function f is linear, or in other words if it has a degree of one and, hence, can be written as $f(x) = ax + b$. On the other hand, if the function f is non-linear, or rather its degree is greater than one, for example the formula $f(x) = ax^2 + bx + c$, the system is said to be non-linear.

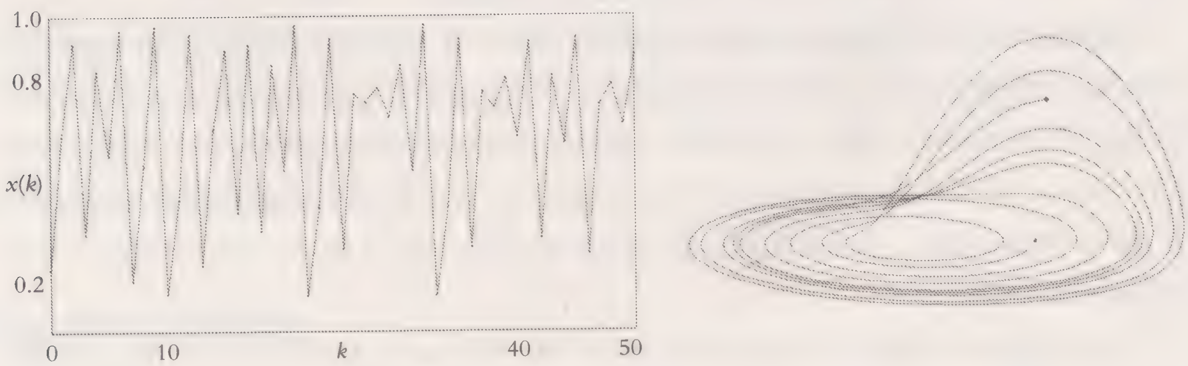
In non-linear dynamic systems, in spite of the fact that the values of the magnitudes that characterise the system are determined by the values at the previous moment in time (the system is said to be deterministic), the output values are not proportional to the input values. Hence, micro-changes in the initial conditions can result in macro-changes in the final states. And this imbalance between cause and effect lies behind the variety of the behaviour they exhibit: they can determine fixed points, periodic orbits, quasi-periodic orbits and, finally, chaotic orbits!



Stationary dynamic systems



Periodic dynamic systems



Chaotic dynamic systems

Types of non-linear dynamic systems (stationary, periodic and chaotic), according to the representation of the time series of values (left) and the graph of the trajectory in the phase diagram (right).

The butterfly effect and the mixing effect

It is now time to answer the question we posed at the start of this chapter: just what is deterministic chaos? Let us first review what we have learnt so far about Poincaré, Smale and Lorenz. We have seen that the geometric essence of chaos consists of stretching and folding trajectories. The successive stretches and folds cause tangles, and cause the phase space to become (imagine each trajectory as like a kind of extremely long string) like a plate of spaghetti Bolognese, where each spaghetti-trajectory is tangled up with the others. Consequently, the slightest imprecision when it comes to measuring the initial conditions means we can make mistakes when it comes to following the correct spaghetti-trajectory, mistakenly following another spaghetti-trajectory tangled up with the one in which we are interested, so that we end up in a far-off region, meaning our long-term prediction will be way off.

'Chaotic' tangle of spaghetti.

To summarise, our historical review of the genesis of chaos theory has taught us there are two structural characteristics related to chaos, both linked to its famous unpredictability. The first is that chaotic systems are highly sensible to initial

conditions (Poincaré, Lorenz), and the second is that solutions to chaotic systems become tangled up and fold in on each other (Poincaré, Smale). Furthermore, we have seen both characteristics arise in Poincaré's three-body problem, Hadamard's billiards, Smale's horseshoe, Lorenz's system, etc.

Hence, the mathematical definition of chaos involves, on the one hand the sensitive dependency on initial conditions, or butterfly effect, and, on the other, topological mixing, or the mixing effect – the trajectories intermingle as if an imaginary baker were kneading them together.

CHAOS = BUTTERFLY EFFECT + MIXING EFFECT

Hence, chaos consists of two types of effect coming together. It is not enough for two trajectories that are close together to diverge rapidly over time, they must also stretch and fold, mixing together. Both effects are the signs of chaos.

There are many famous examples of chaotic systems, the majority of which have already come up in passing. When it comes to continuous dynamic systems, the most representative use of a chaotic system that does not conserve energy (dissipative) is the Lorenz system, which provides a simplified model of the Earth's atmosphere.

For its part, the Hénon-Heiles system, closely related to the three-body problem studied by Poincaré is the typical model for studying chaos in conservative (Hamiltonian) systems. And in the case of discrete dynamic systems, we have already discovered, with the May logistic map – which we shall explain in more detail further on – and the two-dimensional Hénon map, two systems that exhibit Smale-type horseshoes, and, more importantly, symbolic shift dynamics. The symbolic dynamic shift map, or Bernoulli shift map is perhaps the simplest possible example of a chaotic discrete dynamic system.

The shift or 'eraser' map is defined as follows. Given a number x between 0 and 1, expressed as a decimal, $B(x)$ consists of shifting the decimal point one place to the right and erasing the first digit (the whole number, non-decimal part). For example:

$$B(0.324571) = 0.24571.$$

We have shifted the decimal point one place to the right and erased the 3.

Similarly,

$$
\begin{gathered}
B(0.24571) = 0.4571 \\
B(0.4571) = 0.571 \\
B(0.571) = 0.71 \\
B(0.71) = 0.1 \\
B(0.1) = 0 \\
B(0) = 0 \\
B(0) = 0 \\
\ldots
\end{gathered}
$$

The orbit or trajectory of the initial value $x = 0.324571$ is thus $\{0.324571, 0.24571, 0.4571, 0.571, 0.71, 0.1, 0, 0, 0, \ldots\}$. This orbit tends to a fixed point, which is zero (an attractor or sink point).

The Bernoulli map B is chaotic because it exhibits both the butterfly and mixing effect. In terms of the sensitive dependency on the initial conditions, we can carry out the following test. Imagine we are interested in following the trajectory of $x = 1/3 = 0.\widehat{3} = 0.33333\ldots$ however, in our measurement we only make use of a finite quantity of decimal places $y = 0.3333$, causing a small error of less than one-thousandth. To begin with, the orbits of x and y stay close together, but in the long-term they diverge irrevocably:

$$
\begin{array}{lc}
B(0.33333\ldots) = 0.33333\ldots & B(0.3333) = 0.333 \\
B(0.33333\ldots) = 0.33333\ldots & B(0.333) = 0.33 \\
B(0.33333\ldots) = 0.33333\ldots & B(0.33) = 0.3 \\
B(0.33333\ldots) = 0.33333\ldots & B(0.3) = 0 \\
B(0.33333\ldots) = 0.33333\ldots & B(0) = 0 \\
B(0.33333\ldots) = 0.33333\ldots & B(0) = 0 \\
\qquad\qquad\ldots & \ldots
\end{array}
$$

Just like the other periodic decimal numbers, $x = 0.\widehat{3}$ determines a periodic orbit for the shift map. In this specific example, x is a periodic point with a period of 1, or rather a fixed point, since it is not repeated indefinitely. On the other hand, in the same way as for other exact decimal numbers, $y = 0.3333$ is a point that forms part of the sink of attraction of 0, since throughout its orbit it will end up

being attracted by 0. This means that the measurement error, initially less than one-thousandth ($x-y=0.\widehat{3}-0.3333=0.0000\widehat{3}$), will end up disproportionately amplified to the order of tens (after the fourth iteration, the error is equal to $0.\widehat{3}-0=0.\widehat{3}$). Two initial conditions extremely close together give rise to two trajectories, which are completely different after a certain period of time

And what about the mixing effect? To check for its presence, we shall use infinite, not periodic decimals, or rather irrational numbers. Consider the orbits of $\sqrt{2}-1$ ($=0.41421356237\ldots$) and of $\pi-3$ ($=0.14159265358\ldots$):

$B(\sqrt{2}-1)=0.14213\ldots$	$B(\pi-3)=0.41592\ldots$
$B(0.14213\ldots)=0.42135\ldots$	$B(0.41592\ldots)=0.15926\ldots$
$B(0.42135\ldots)=0.21356\ldots$	$B(0.15926\ldots)=0.59265\ldots$
$B(0.21356\ldots)=0.13562\ldots$	$B(0.59265\ldots)=0.92653\ldots$
$B(0.13562\ldots)=0.35623\ldots$	$B(0.92653\ldots)=0.26535\ldots$
$B(0.35623\ldots)=0.56237\ldots$	$B(0.26535\ldots)=0.65358\ldots$
...	...

What do we see? The decimal numbers obtained are completely random! It is as if they had been picked by a lottery machine. This is the chance that gives rise to chaos. Both the orbit of $\sqrt{2}-1$ and that of $\pi-3$, and in general the orbit for any irrational number, will ricochet between 0 and 1. It will just as quickly lurch from being close to 0 and close to 1 (or close to 0.5). To summarise, the decimal places of an irrational number do not conform to a recognisable pattern. Hence, while rational numbers – exact, periodic decimals – give rise to orbits that sooner or later become periodic (that are repeated, albeit infinitely), irrational numbers, on the other hand, are non-periodic, infinite decimals that determine completely irregular and erratic orbits. Since all rational numbers are infinitely close to irrational numbers and vice versa, the periodic and aperiodic orbits mix irrevocably. This is the mixing effect.

However, we might ask where we can find the famous stretch-and-fold operations that give rise to chaos. Here we must pay careful attention to the mathematical operations we are carrying out when we apply the Bernoulli map B. We have said that this map consists of shifting the decimal place one place to the right and erasing the first digit. In fact, by shifting the decimal point, what we are actually doing is multiplying the number by 10, or in other words 'stretching' it, and by removing

RETURNING TO BERNOULLI'S SHIFT MAP

The symbolic dynamic shift map has other extremely important properties:

1) It cannot be computed on a computer. Since computers work with a limited number of decimal digits (with a finite precision), they represent all numbers as exact decimals and, hence, if we program Bernoulli's map on the screen, all the orbits – just like exact decimals – will be attracted by the point 0. Not a hint of chaos.

2) There exist periodic orbits with any period. Just as there are decimal numbers with any period (e.g. period six: $0.\overparen{346235}$), there are also orbits with every imaginable period: 1, 2, 3, 4, 5… In this respect, there is a famous theorem by the mathematicians Tien-Yien Li and James Yorke, based on another by Sharkovsky, which states that given a continuous function, if there is an orbit with period three, there are orbits with any period. This is indeed the case. The existence of a 3-cycle implies the existence of an n-cycle (for $n = 1, 2, 3, 4, 5…$). Li and Yorke provided an excellent summary in the title of their article: *Period Three Implies Chaos.*

3) The symbolic dynamic shift map is, as Hadamard and Smale discovered, one of the most easily recognisable signs of chaos. Smale's solenoid and horseshoe and the Lorenz attractor exhibit symbolic shift map dynamics. If, instead of considering decimal expressions using base 10, we make use of base 2 (i.e. using just ones and zeros), each trajectory of the Lorenz attractor can be identified by a sequence of zeros and ones. For example, the trajectory 0.11000101… indicates the trajectory will first rotate twice around the right eye of the attractor (there are two 1s in a row after the decimal point) then three times around the left eye (the two 1s are then followed by three 0s in a row), etc. Using this symbolic dynamic, it is possible to show the existence of chaos in Lorenz's system, since the trajectory alternates erratically around each attractor.

the first digit we are actually 'folding' it. Once again, the magic recipe for chaos.

Let us now consider an application of the May logistic map, described by the following difference equation.

$$x_{n+1} = kx_n(1 - x_n).$$

In other words, given an initial condition x between 0 and 1, the orbit of x is calculated by repeatedly applying the function $f(x) = kx(1-x)$, where k is a parameter greater than 1 but less than 4. In fact the behaviour of the logistical system (so called because it can be used to model the dynamic of certain populations) exhibits a spectacular dependency on the value assigned to k. If k is below a certain critical value, estimated as 3.569945…, the orbits or trajectories are quite regular. However, once this limit has been passed, they diverge irrevocably towards chaos. This discrete dynamic system provides a clear example of how complex properties appear behind simple mathematical operations. Let's check.

The first thing we need to do is check that $f(x)$ is a second-degree function:

$$f(x) = kx(1-x) = kx - kx^2.$$

In other words, $f(x)$ is a non-linear function and it is precisely this non-linearity that makes possible all sorts of chaotic behaviour, since only in this way can small causes have large effects.

Let us begin by studying the dynamic of the logistic map when k is below the critical value, for example $k = 2$. We take $x_0 = 0.8$ as the initial condition and use a pocket calculator to calculate its orbit:

$$x_1 = f(x_0) = 2x_0(1-x_0) = 2\cdot 0.8\cdot(1-0.8) = 2\cdot 0.8\cdot 0.2 = 0.32$$
$$x_2 = f(x_1) = 2x_1(1-x_1) = 2\cdot 0.32\cdot(1-0.32) = 2\cdot 0.32\cdot 0.68 = 0.4352$$
$$x_3 = f(x_2) = 2x_2(1-x_2) = 2\cdot 0.4352\cdot(1-0.4352) = 2\cdot 0.4352\cdot 0.5648 = 0.49160192.$$

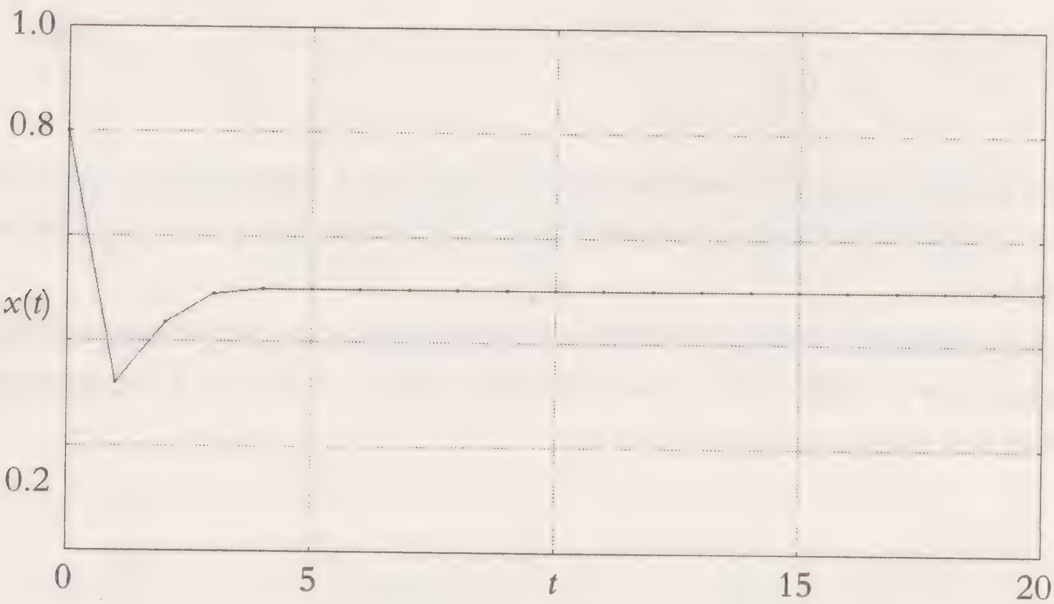
Now we know how to calculate the first terms of the orbit, we can calculate the following terms directly:

$$x_4 = 0.4998589\ldots$$
$$x_5 = 0.4999998\ldots$$
$$x_6 = 0.4999999\ldots$$
$$\ldots$$

By examining the numbers, we can see what happens. They grow increasingly close to 0.5. Our trajectory clearly approaches a limit, which is the attractor point 0.5. Out of curiosity, let us calculate the orbit of 0.5. Since

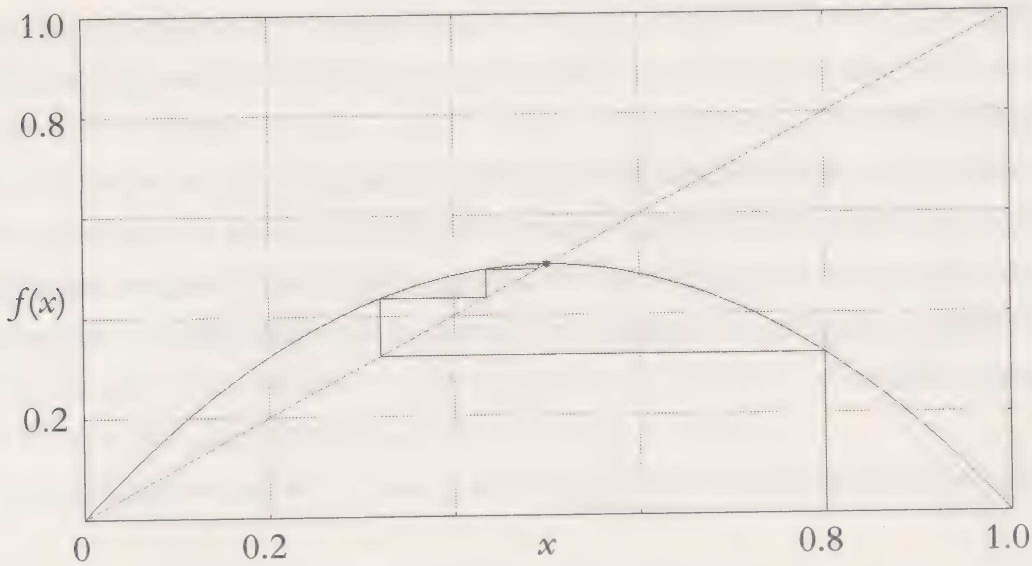
$f(0.5)=2 \cdot 0.5 \cdot (1-0.5) = = 2 \cdot 0.5 \cdot 0.5 = 0.5$, we can see that the orbit for 0.5 is stationary (it is always equal to 0.5). Hence, the orbit for 0.8 converges on a point of equilibrium.

However, let us observe how, from a geometric perspective, our trajectory tends towards this fixed point. To do so, we shall use two methods. In the first, we use a computer program to represent how the values of the orbit evolve (represented on the vertical axis), as the number of iterations increases (represented on the horizontal axis):



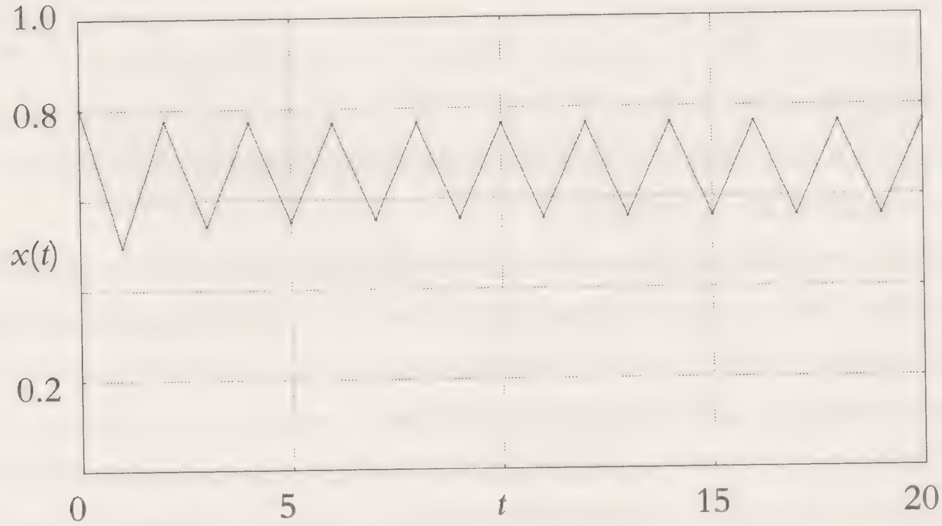
We can see that the values of the orbit soon stabilise around 0.5, something we already knew thanks to the calculator.

In the second, we shall make use of a 'cobweb diagram', which allows us to draw the orbit of a point. Once we have represented $f(x)=2x(1-x)$, which gives a parabola because $f(x)$ is a second-degree function, we take our initial condition as $x_0=0.8$ and we set out to calculate its orbit graphically. We draw a vertical line from the x axis $x_0=0.8$ until it intersects with the parabola $f(x)$. We then draw a horizontal line to the point at which it intersects the parabola until it reaches the diagonal $y=x$. The new x coordinate is hence given by the point at which the line intersects with the diagonal and corresponds to x_1. We then move vertically – upwards or downwards – until once again intersecting the graph of $f(x)$. Repeating the process yields a broken line whose vertical segments have x coordinates $x_0, x_1, x_2, x_3\ldots$ and indicates the point to which the orbit of the initial condition x_0 tends:



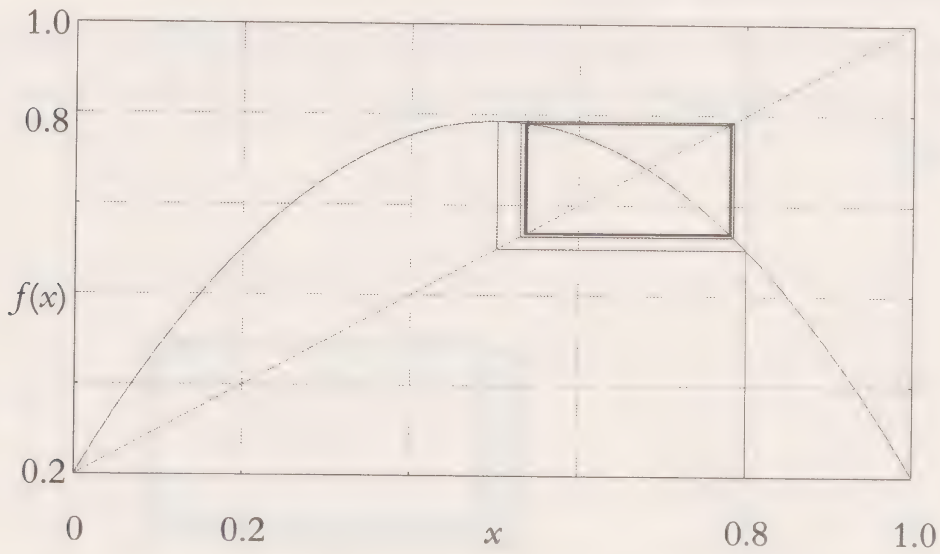
In this graph, we can see how the 'cobweb' of $x_0 = 0.8$ converges on a fixed point, the point that intersects the parabola $f(x)$ and the identity line $y = x$. As we expected, it is 0.5.

Let us repeat the previous study, modifying the value of the parameter k. Instead of assigning it the value of 2, let us assign it the value 3.1. For $k = 3.1$, this makes the orbit of our initial condition $x_0 = 0.8$:

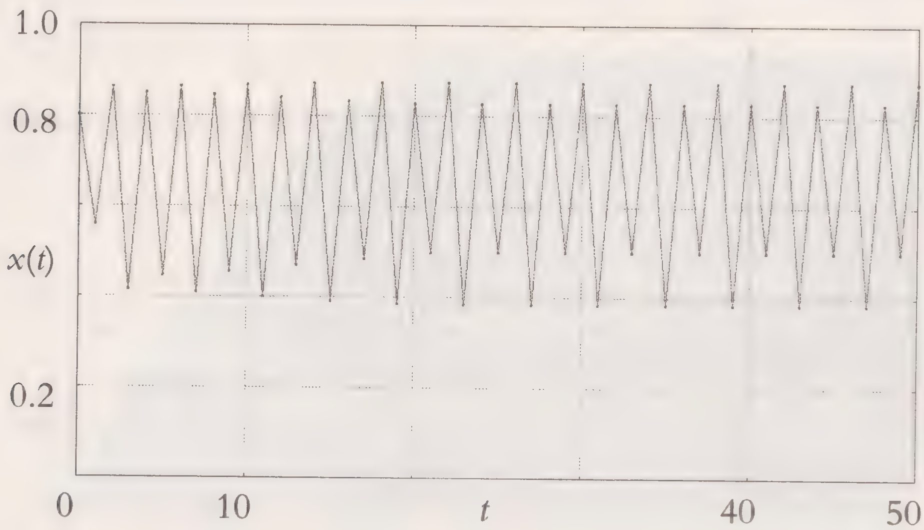


Something strange happens when k is greater than 3. Although the movement continues to be regular, there is no longer a single limit point to which the orbit of 0.8 tends. Instead, it now tends to oscillate between two values, 0.56 and 0.76, as can be seen in the graph. It is as if the attractor point 0.5 has split into two attractor

points: 0.56 and 0.76. We are clearly faced with a period 2 or 2-cycle orbit, since there are two attractor points. This is the new cobweb, which, as we can see, no longer yields a point, but a rectangle:



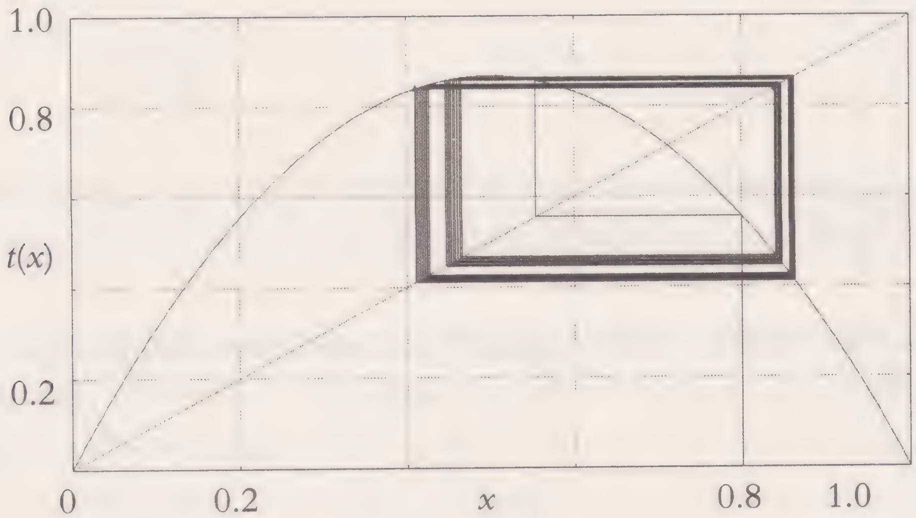
Let us now continue increasing k and consider the case $k=3.5$. This is the orbit of $x_0=0.8$:



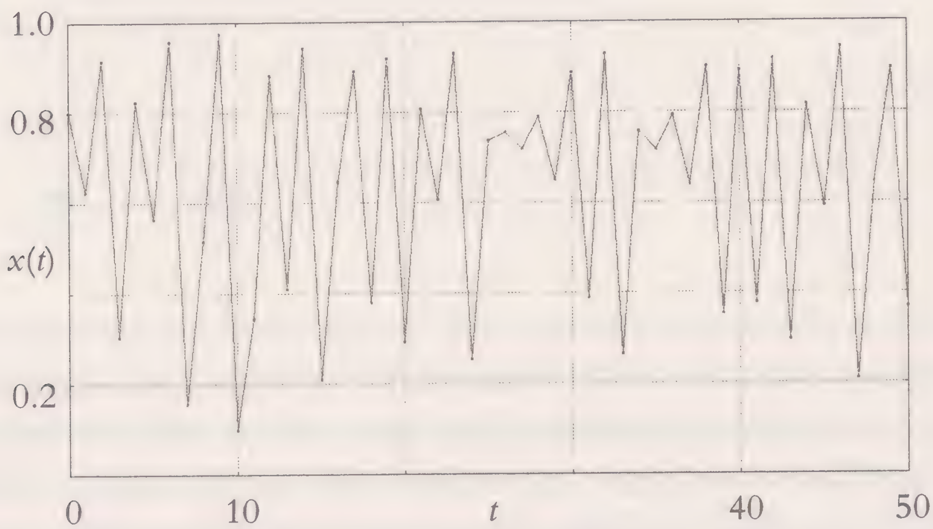
The orbit now oscillates between four points, which are approximately 0.39, 0.51, 0.82 and 0.86. If, for $k=3.1$ there were two attractor points, for $k=3.5$ there are now four. Moreover, instead of a periodic orbit with period 2, we obtain an orbit with period 4, or a 4-cycle (the values are repeated every four steps or iterations). It seems that as we increase the value of k, the periods double (i.e. they are multiplied

by two: 1, 2, 4...). To begin with, there was a single attractor point, then there were two attractor points, now we have four, and we will soon (as we can imagine) have eight, sixteen, thirty-two, etc. The dynamic is no longer so simple, although the pattern remains more or less regular.

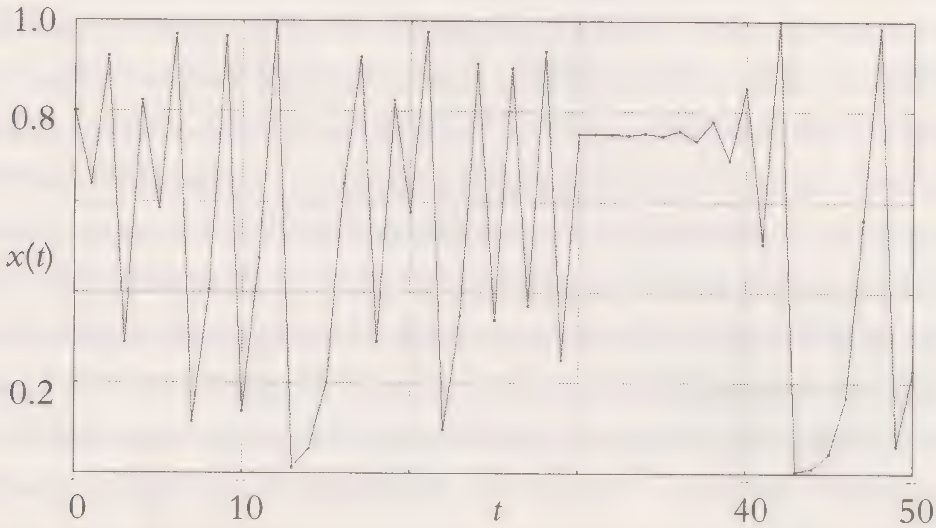
We shall return to this strange phenomenon of the doubling of the period later on. For the time being, let us settle for drawing the new cobweb (made up of two rectangles):



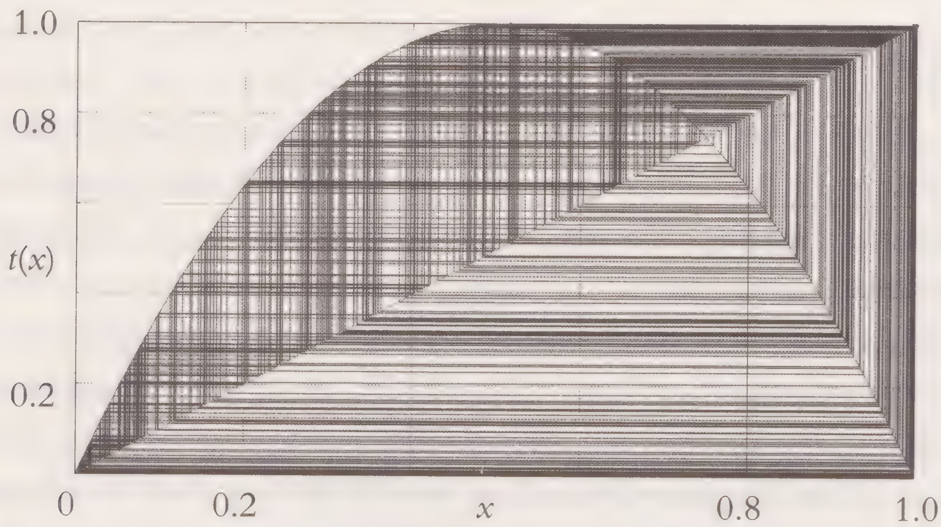
Finally, let us go beyond the critical value of 3.569945, letting $k = 3.9$. The situation changes drastically. Here is the orbit for $x_0 = 0.8$:



The orbit has become chaotic! It does not exhibit any regular pattern. It is not even quasi-periodic. It jumps from one point to another; it appears to be random. And if we let $k = 4$?



The same chaotic behaviour! And, similar to the orbit, the cobweb is also chaotic since it moves erratically around inside the interval 0 and 1:



However, the orbit and the cobweb of $x_0 = 0.8$ are not exceptions, since all the other possible orbits and cobwebs do exactly the same thing, mixing amongst each other. Once again, we find the mixing effect.

The surprises do not stop here: two different initial conditions, despite being extremely close together, define orbits which, after a certain period of time, are

completely different. Hence, taking $k = 4$, if we consider the evolution of the initial point $a = 0.900$ and an error in the initial measurement, $b = 0.901$ (our measurement contains an error of the order of one-thousandth), we can see how the orbits of a and b visibly diverge sooner rather than later, despite being close together at the start. In fact, using the recurrence, the orbit of a is $\{0.900, 0.360, 0.9216, 0.2890, 0.8219, 0.5854, 0.9708...\}$ and the orbit of b is $\{0.901, 0.3568, 0.9180, 0.3012, 0.8419, 0.5324, 0.9958...\}$. Or rather, we can see how the initial difference of one-thousandth can increase considerably to the order of hundredths after a few iterations. In fact, it has been multiplied by 20 in just seven iterations! This is responsible for the fact that the real and predicted trajectories are completely different after a certain period. Once again, we come up against the butterfly effect.

To summarise, by repeatedly changing the values of the parameter k for the logistic map from $k = 2$ all the way to $k = 4$, we have seen how the system gradually falls into the depths of chaos. However, where are the stretching and folding operations that produce the chaos? Right before our eyes. The logistical function $f(x) = kx(1-x)$ stretches the interval of numbers between 0 and 1 as a result of multiplying x by k, and then folds it over itself as a result of multiplying the result of kx by $(1-x)$, a value smaller than the one. Stretching and folding in the shape of a horseshoe.

In search of chaos

To this day, although there is no canonical definition among mathematicians, there is a consensus that deterministic chaos is the combination of two effects – the butterfly effect and the mixing effect – which we have seen both in the Bernoulli shift map and the May logistic map.

However, in which type of dynamic systems can we expect chaotic behaviour? Where should we look for it if we really want to find it? To begin with, as we already know, in non-linear systems, since only in these systems can the effect of a sum of causes not be the sum of the effects of each of the causes taken separately, but something much more explosive and sensational. Secondly, and this is the new element, in non-integrable systems. An integrable system is a system in which the trajectories or solutions can be expressed using known functions. In other words, the term integrable refers to the possibility of giving the solution a specific expression; in summary – a formula. Integrable systems (regardless of whether they are linear or not) are predictable, since they have a formula that allows us to directly calculate the

orbit of any point at any moment in time. On the other hand, for non-integrable systems, the solution cannot be obtained using a formula, meaning that it is not possible to make predictions in time all the way to infinity. Furthermore, from the topological perspective of Poincaré, we can see that when a system is non-integrable, this is because its trajectories are extremely tangled.

Crossing these two categories leads us down an important path. Non-linear and non-integrable systems can exhibit an irregular and unpredictable behaviour that indicates the presence of chaos. However, it should be pointed out that even when chaos requires non-linearity (so that small changes can produce large changes) and non-integrability (so that it is not possible to make long-term predictions), a non-linear and non-integrable dynamic need not necessarily be chaotic. There are non-linear and non-integrable systems that exhibit regular and predictable motion. These two properties (of non-linearity and non-integrability) are – in mathematical language – necessary but not sufficient.

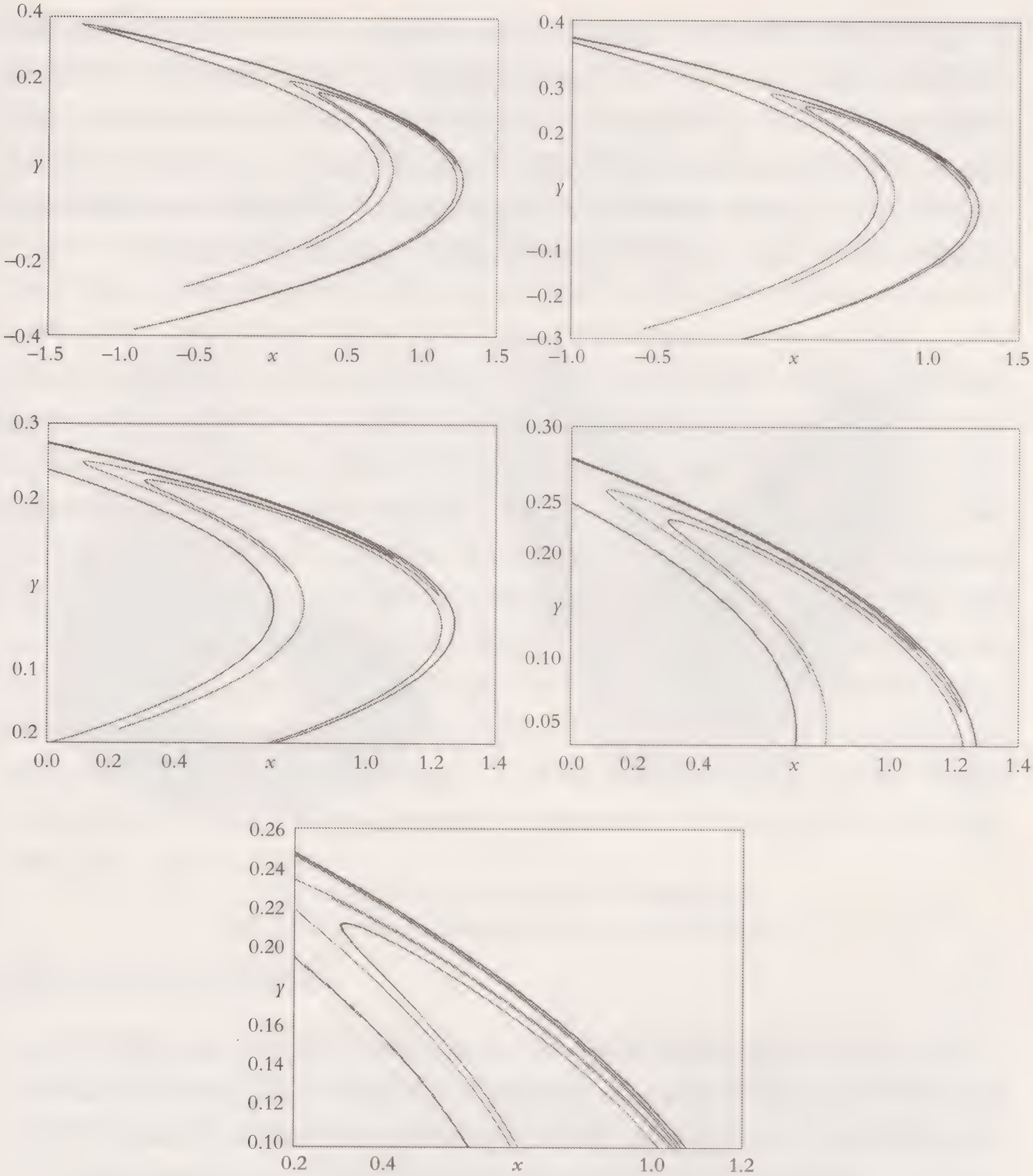
Furthermore, among the non-linear systems and non-integrable systems, which are, let us repeat, the only candidates for being chaotic systems, it is possible to distinguish two different types: Hamiltonian systems, which conserve energy, and dissipative systems, which do not conserve energy. These two types of systems will provide us with the two types of deterministic chaos that are known today.

Hamiltonian chaos occurs in systems that conserve energy, such as the three-body systems studied by Poincaré, the stellar system studied by Hénon and Heiles, and Hadamard and Sinái's billiards. This chaotic behaviour has, as we have seen in Chapters 1 and 2, a homoclinic origin, due to the infinite number of intersections that take place between the separatrices of the saddle point, causing a monumental tangle of trajectories. However, these systems, despite exhibiting extremely complex dynamics, lack the slightest trace of the presence of 'strange' or 'pathological' attractors. Indeed, there is a famous mathematical theorem, the Liouville theorem, that states that the conservation of energy prevents attractors from appearing, since they are dissipative structures in which energy is lost as the system is drawn towards the attractor.

In contrast to this, non-Hamiltonian chaos – often referred to as pure chaos – is the chaos that arises in systems that do not conserve energy, such as the Lorenz system. Since they do not conserve energy, these systems exhibit attractors that give rise to some of the best-known chaotic configurations – strange attractors, which represent the bridge between chaos theory and fractal geometry.

A strange attractor is an attractor of a chaotic system with fractal geometry. And a fractal is a rough, irregular geometric object, with infinite details, self-similarity and whose dimension is probably a fraction. Strange attractors are complex structures, which, if repeatedly expanded, exhibit the self-similarity of fractals. The same structure always appears over and over again. The parts are similar to the whole. Furthermore, in many cases their dimension is a fraction. In other words, if we are on the plane, the dimension of our fractal attractor will be a number greater than one but less than two, such as one and a half. It will occupy more than a curve but less than a surface. And if we are dealing with space, its dimension will be greater than two, but less than three, for example two and a quarter. It will occupy more than a surface but less than a body. This means its dimension is fractal and not a whole number. For example, the dimension of the Lorenz attractor is around 2.06. Curiously, in spite of the fact that its discovery has been definitively recognised, the 'strange' property of the Lorenz attractor (in other words its condition as the attractor set of a chaotic system, with probable fractal geometry) had yet to be mathematically proven in 2000. In fact, in 1998 Stephen Smale listed its proof as one of the open mathematical problems of the 21st century. In 2002 the mathematician Warwick Tucker managed to provide a fully rigorous proof of its existence in an article entitled 'The Lorenz Attractor Exists'. The butterfly-shaped attractor that appeared on Lorenz's computer screen was not fiction, it is real. The same thing happened with the Hénon strange attractor, discovered with the help of a computer in 1976, but whose existence was not mathematically proven until 1987 by the Swedish mathematician Lennart Carleson (Abel Prize winner in 2006).

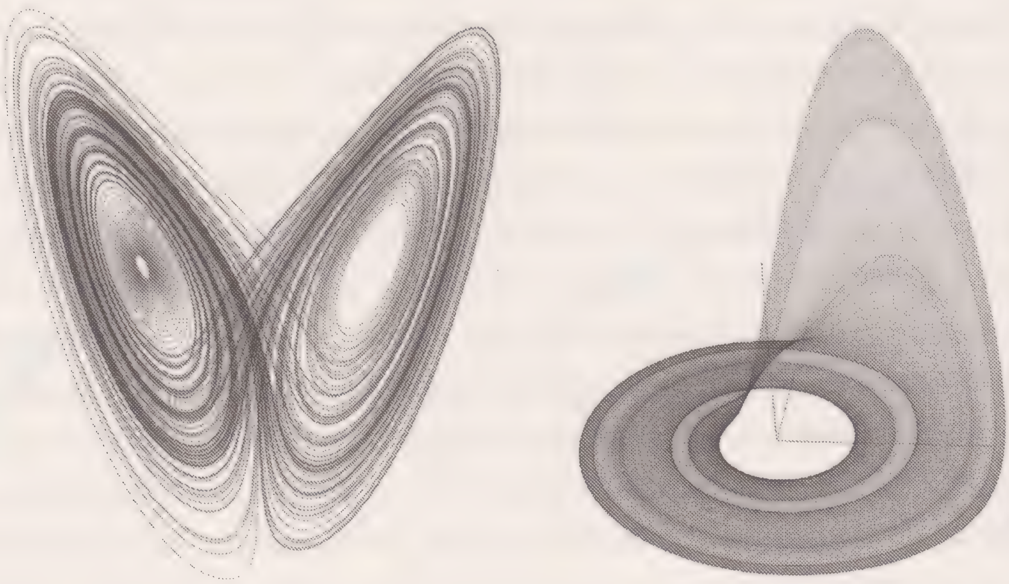
The Ueda strange attractor. This attractor, which looks like a whirlpool, is the Poincaré section of a chaotic system.



From left to right and top to bottom, successive amplifications of the Hénon attractor. All these exhibit the same pattern (rays that unfold into more rays).

The Rössler attractor, on the other hand, has not fared so well. Otto Rössler proposed a series of equations that describe the behaviour of the Belousov–Zhabotinsky chemical reaction, an oscillating reaction in which compounds continuously join and separate, producing a series of spectacular red-blue coloured

changes. The computational simulation of the solutions to the system of differential equations exhibited a chaotic behaviour similar to the one studied by Lorenz in his system. Furthermore, this suggested – heuristically, as in the case of Lorenz – the presence of a strange attractor, the Rössler strange attractor, whose existence remains unproven. As yet nobody knows if it is real, if an attractor is hidden beneath the tangle of trajectories, or if it is just a computer illusion… generated by *Matrix*.



Lorenz (left) and Rössler (right) strange attractors.
Mathematicians have yet to prove the existence of the latter.

And what are the implications of attractors with fractal geometry for the dynamic? We may be forgiven for thinking that there are none; after all, what does speed have to do with the price of fish? However, we would be wrong. If there is one thing to be learnt from Poincaré, Smale and Lorenz, it is that the dynamic is always based on geometry.

In classical attractors (with fixed points and limit cycles) neighbouring orbits remain close, and the small errors (as Laplace expected) remain bounded, making it possible to make long-term predictions. On the other hand, strange attractors, the ones related to chaos theory, are substantially different. Two orbits with similar initial conditions diverge extremely quickly and only remain close together for a short period of time. If, in order to imagine what happens to the neighbouring trajectories in a strange attractor, we add a small drop of dye to trace their action, we will see

how the drop becomes an extremely long and fine thread, threaded throughout the whole attractor. In spite of the fact that the points we have marked may be extremely close at the beginning, they will end up scattered all over the attractor. A definite prediction of the final state of any of these points is impossible if there is the presence of the slightest error in the measurement, since the final state can lie in any region of the strange attractor. Chaos confuses the orbits in the same way that a baker mixes dough when they knead it. The key, as always, lies in geometry, in the stretch and fold operations. The orbits must be stretched, increasing the errors (the butterfly effect) but they must also be folded in on each other, tangling them up in the space towards the attractor (the mixing effect). The stretches increase the uncertainties, whereas folds, by bringing together trajectories that are initially separate, destroy the initial information. The trajectories mix amongst each other, like cards in the hands of a gambler. However, when the process of stretching and folding is repeated *ad infinitum*, the attractors of chaotic systems tend to exhibit more and more folds within each fold. This makes the geometry of chaotic attractors much more complex than their classical siblings. They are objects that gradually reveal more and more details as the image is expanded. In fact, they exhibit an infinite level of detail, with one fundamental property – they are self-similar. That means that their microscopic structure exhibits the same complexity as their macroscopic structure. They are, in a word, fractals.

Small scale examples

Admitting the existence of mathematical systems with a chaotic dynamic, what is the real relevance of this phenomenon? Are chaotic systems pathological cases or, on the other hand, are they important in the physical world? Is chaos the exception or the rule? In which physical systems can we find mathematical chaos?

Chaos is ubiquitous. It appears everywhere, both in the motion of celestial bodies (the three-body problem) and the behaviour of a double pendulum, in fluids on the brink of turbulence (Rayleigh–Bénard flow), in certain chemical reactions (the Belousov–Zhabotinsky reaction), and in certain biological populations. Moreover, the discovery of the ubiquity of chaos can be regarded as the third great revolution in science from the previous century, after relativity and quantum mechanics.

For example, within the Solar System, one noteworthy chaotic movement is the shaky movement of Hyperion, one of Saturn's moons, whose potato shape results in

an apparently fortuitous trajectory. The satellite has a regular orbit around Saturn, but its path is in fact disordered. It exhibits fast chaotic motion that manifests itself as a lack of precision in the position of the satellite over an interval of six hours. Quite literally, Hyperion goes tumbling through its orbit.

Furthermore, in 1988 two scientists from MIT, G. Sussman and J. Wisdom, presented numeric evidence that the motion of Pluto is also chaotic. In fact, the trajectory of Pluto is of particular interest since its orbit crosses that of Neptune

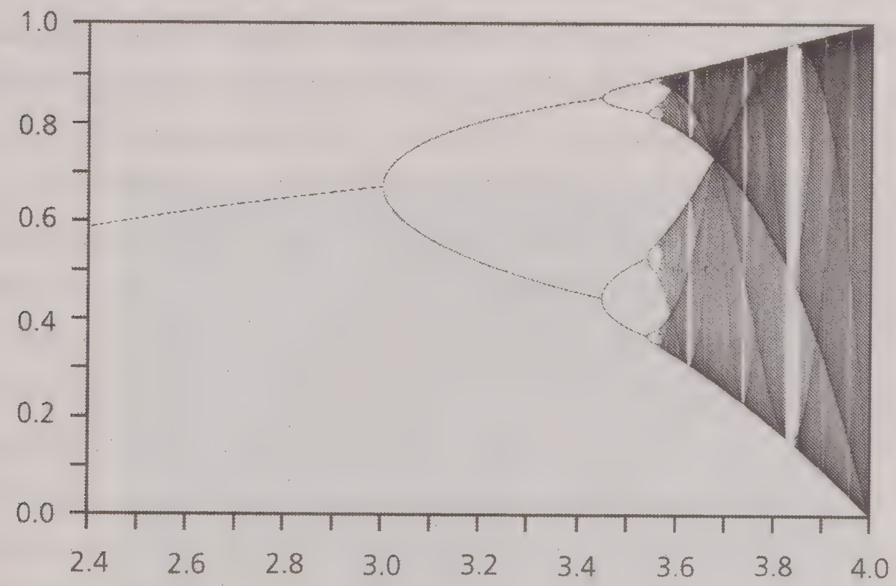
SEARCHING FOR CHAOS WITH FEIGENBAUM

Mitchell Feigenbaum (1944–) is a mathematical physicist and a pioneer in computer-aided chaos research. In 1975, playing with his computer he discovered a certain number by trial and error, now referred to as the 'Feigenbaum constant' or 'chaotic constant', which characterises the transition from periodic to chaotic motion. In fact, we have already observed this surprising phenomenon in our study of the logistic map. As we gradually alter the value of the parameter k, on which it depends, the periods of the orbit double, or in other words are multiplied by two. This means we go from orbits with period 1 to orbits with period 2, 4, 8, 16, 32, etc., until reaching chaos after passing the critical value of k, in other words 3.569945.... The propagation of the doubling of the period from $k=2$ until this value is so quick that the period ends up being doubled an infinite number of times, giving way to chaos. As k increases, so too does the complexity of the logistic system: stationary → periodic → chaotic. Representing the point or points on which the orbit of $x=0.8$ converges using the logistic map for different values of the parameter k gives the diagram on the opposite page. In this graph, the values of k are represented on the horizontal axis, while the values to which the orbit $x=0.8$ tends are represented on the vertical axis. Specifying a value of k, the vertical line provides an image of the corresponding attractor in the interval between 0 and 1. For example, for $k=3.0$ the vertical line only cuts the graph at one point, meaning that we are dealing with a point with period 1, a fixed point. However, for $k=3.2$, the vertical line cuts the graph and two points, and hence our orbit has 2 cycles. Indeed, as we move horizontally along, from $k=2.4$ to $k=4$, the branches of the Feigenbaum tree continue to bifurcate as a result of the doubling of the period. Passing the critical value of 3.569945... (the point of chaos), the attractor, which is defined by each vertical line, turns into a chaotic band. In fact, it is a fractal (a Cantor set). There are, however, intervals of periodicity once this limit has been passed. For example, after 3.82, there is a strip of the diagram for which only the points of intersection with an imaginary vertical line can be seen, one at the top, another in

and it might be the case that sometime in the not too distant future they are close enough to mutually disturb each other, triggering a cosmic catastrophe. Sussman and Wisdom used a supercomputer to calculate the trajectory of Pluto for the next 845 million years and discovered that, due to the uncertainty of the initial conditions, two of the variables, which were initially close together, determined two trajectories that diverged considerably over an interval of just 20 million years – only a short time if we consider that the Solar System is estimated to be at least 4.5 billion years old.

the middle and another at the bottom, meaning our orbit now has 3 cycles. Furthermore, as we shall see, since a period of three implies chaos, we should not be surprised by what comes next – a new chaotic jumble of points.

Feigenbaum calculated the quotients of the relative distances between bifurcations – to make things clearer, between the sizes of the bifurcations of the tree – and saw that the limit value was 4.669201… That is independent of the map being used – regardless of whether logistic or any other – and is universal. Although Feigenbaum discovered his constant by means of a practical or heuristic method, and not via a formal proof, his discovery was – and still is – regarded as brilliant.



Bifurcation or Feigenbaum diagram for the logistical map.

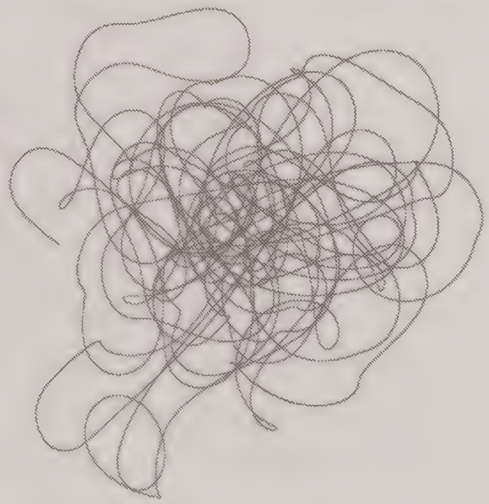
Fortunately, for our planet, a slower chaos applies: the imprecision in the position of the Earth appears at an interval of 100 million years.

Hyperion, Saturn's irregular satellite, photographed by the Cassini–Huygens craft.

However, there are more examples that show how chaos has moulded the Solar System in its image and similarity. For example, the asteroid belt that lies between Mars and Jupiter moves under the action of the Sun, but is disturbed by perturbations caused by Jupiter. Hence it constitutes a three-body system (Sun + Jupiter + asteroids). In this system, certain movements are regular, whereas others are chaotic. The asteroids that move with regularity remain in their orbits, but those that follow chaotic trajectories end up escaping and become lost in space after a while. Therefore, the distribution of the asteroids in the belt is not uniform. There are bands in which there is nothing – the Kirkwood gaps, named after the US astronomer who discovered them around 1860. If, in its orbit around the Sun, an asteroid crosses one of the zones prohibited by Kirkwood, its period of revolution will 'resonate' with that of Jupiter, and the gas giant will escape from its orbit, possibly towards Mars, or worse still, towards the Earth, ending the apparent harmony we observe in the Solar System. Curiously, a similar phenomenon occurs with the empty bands in Saturn's rings, where the particles that orbit in the zones of resonance escape and gaps form.

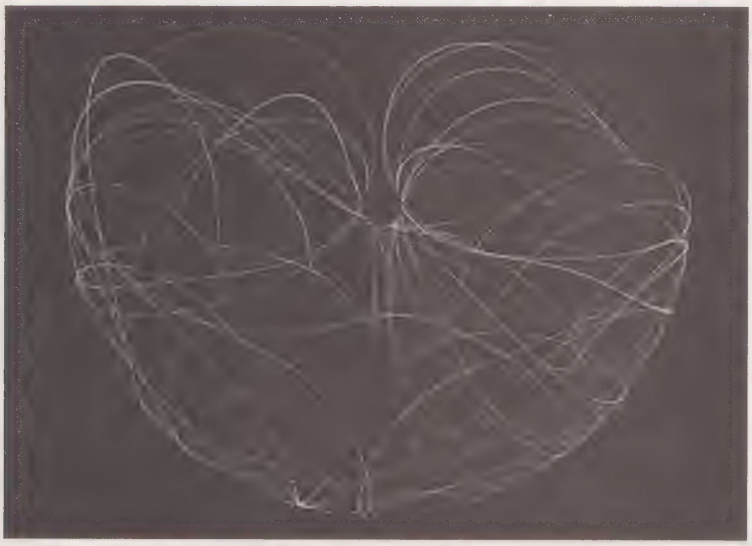
AN ANTI-NEWTONIAN WORLD

The US physicist Julien C. Sprott (1942–) has imagined a world parallel to our own, in which Newton's first two laws hold, but the third – the law of action and reaction – does not. In this strange world, the magnitude of the forces between two bodies that interact are not equal in magnitude and in opposite directions, but have the same magnitude and the same direction. This means that when a frog resting on the leaf of a water lily jumps, it does not push the leaf backwards, but forwards, dragging it with it. This gives rise to a dynamic that exhibits some extremely curious and unfamiliar features, including chaos within the two-body problem. This is the anti-Newtonian dynamic.



Chaotic orbit in the anti-Newtonian two-body problem.

In fact, the most surprising thing of all is not that multiple complex systems – such as the Solar System or the weather, the climate and the atmosphere, to which we shall return in subsequent chapters – exhibit chaos, but that this is also the case for extremely simple systems, such as a pendulum. In fact, if we consider a double pendulum, which



Chaotic motion defined by a double pendulum.

THE TAP HAS NOT BEEN TURNED OFF PROPERLY

Most of us will have seen water dripping from a tap that has not been turned off properly. However not everyone knows that chaotic behaviour lies behind this phenomenon. In fact, on many occasions the pause between drops is not regular, but aperiodic and unpredictable. In one word: chaotic.

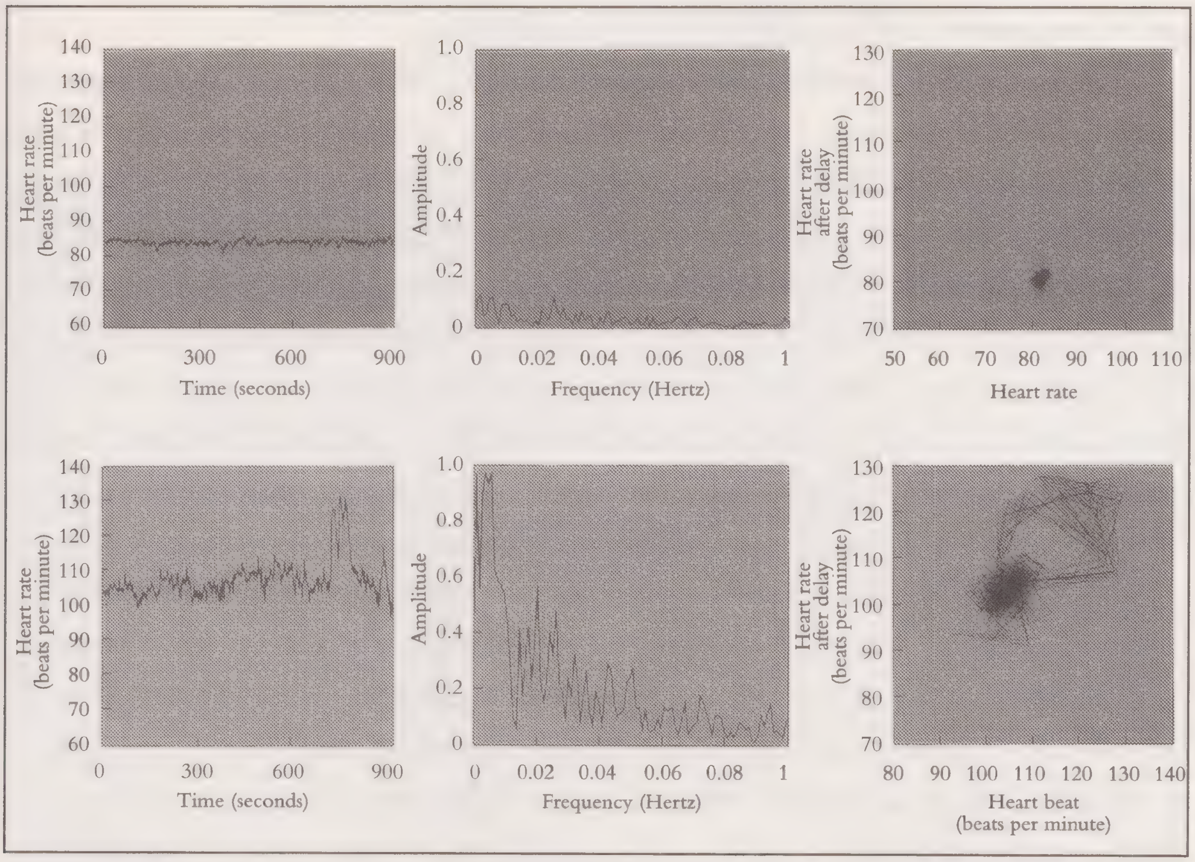
The phenomenon was studied by Robert Shaw, together with other scientists at the University of California. The experiment began by using a microphone to measure the intervals of time between drops. After obtaining the series of time intervals, the values were grouped in pairs, giving a sequence of pairs of numbers, or points on the plane. By representing them graphically, the researchers were able to obtain a section of the underlying attractor. When the rate of the drops was periodic, a sort of limit cycle appeared. However, when the rhythm became random, a sort of strange attractor appeared. This did not take the form of a mark, but instead was a structure with certain intrinsic geometry. The shape of a horseshoe, the simplest signature of the process of stretching and folding that gives rise to chaos. The randomness was supported on a deterministic 'scaffold'.

consists of a pendulum with another suspended from its end, we can see that over a certain energy level, the motion of the system becomes chaotic. When we are dealing with a single pendulum, the motion is extremely simple, but when we join two pendulums together, the oscillations become completely unpredictable.

Large-scale applications

In recent years, the influence of chaos theory, non-linear dynamics and the sciences of complexity in general have provided considerable stimulus to the life sciences (e.g. biology, medicine etc). In fact, the collaboration between 'hard' and 'soft' sciences has given rise to a highly productive symbiosis in the space of just a few years. If we cast our minds back, we shall find a good example. Until well into the 20th century, medicine and physics appeared to be separated by a solid and impenetrable wall, except for the eruption of radiation when it came to treating and diagnosing cancer. However, from the 1950s onwards, the wall became increasingly porous, and we have all benefited as a result. To give just one example, medical images have been made possible thanks to the joint development of mathematics, physics and medicine.

Similarly, chaos theory has formed a link between developments that are, in principle, far removed from reality to become powerful tools in the hands of experts who use the same language as doctors. In this respect, the medical application of chaos theory does not make it possible to precisely predict or solve individual problems, but rather 'characterise' (using certain 'magic numbers', such as Lyapunov exponents and fractal dimensions) certain aspects of the behaviour of systems as complex as biological systems. Put another way: chaos theory can help to classify states. Its merit arguably does not lie in the value of the number that is obtained, but in the reformulation of medical problems from an observational paradigm, to one that allows them to be modelled and measured. Cardiology, electro-encephalogy and magneto electro-encephalogy are good examples in this respect. In just a few years, the studies of chaos and fractals in physiology have been able to provide more sensitive methods for characterising, for example, the dysfunction caused by ageing or illness. In summary, the great discovery is as follows – human beings are complex and chaotic when we are healthy, and rigidly ordered when ill.

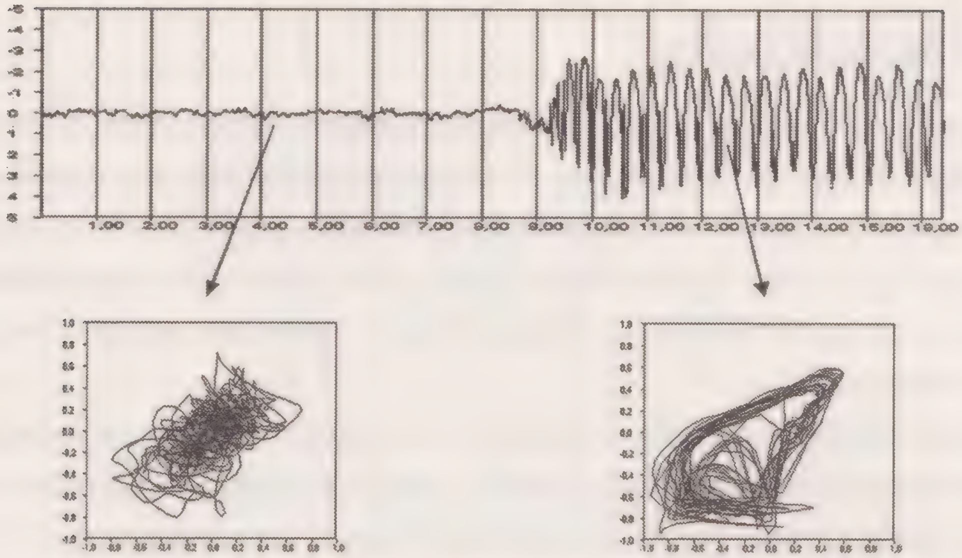


Different measurements of the heartbeat of an ill patient (above) and a healthy one (below). When the activity is periodic and predictable, it indicates a cardiopathy, whereas when the individual is healthy, the measurements are chaotic.

Experimentally, the problem lies in using the series of data observed or measured during the time interval (e.g. the heart beat or brain wave) to reconstruct the evolution of the dynamic system (the heart or brain) in the phase space where we can accurately measure and calculate the magic numbers of chaos – the Lyapunov exponents, the fractal dimension… It is at this point that a highly ingenious trick devised by David Ruelle and Floris Takens comes to our aid. It involves using delayed values to reconstruct the attractor of the system in some way. Given the series of data $x_1, x_2, x_3, x_4,\ldots$ it is possible to form a set of pairs (x_1, x_2), (x_2, x_3), $(x_3, x_4)\ldots$ These points determine a trajectory on a plane. If we group the data into threes, we obtain a trajectory in space. Hence, the dynamic of our system is represented by the dynamic of this set of points, and it is now possible to calculate the fractal dimension or the Lyapunov exponents. There is a mathematical theorem that states that the longer the delay used for reconstruction (in other words, instead of grouping the data into twos or threes, we do so in fours or fives), when the system is periodic, the fractal dimension increases to a given value and then remains constant and whole (hence it is not a fraction, or a fractal). On the other hand, if the system is chaotic, the fractal dimension stabilises around a fraction and at least one of the Lyapunov exponents is positive.

However, is all this mathematical apparatus really useful? Like it or not, the answer is yes. Generally speaking, we can state that simple dynamic systems imply illness, whereas complex (chaotic) dynamic systems imply health, paradoxical as it might seem. Hence, illness involves a loss of complexity and any increase in regularity brings us closer to death. In fact, the appearance of regularities in the heartbeats or brain pulses of critical patients is a bad sign. Using electrodes to measure the electric brain signals, the curve resulting from this analysis first appears chaotic (aperiodic) and fractal (self-similar, creased). Applying the Ruelle–Takens method for reconstructing the attractor using delays, we can see that healthy patients display strange attractors, whereas patients with encephalopathies exhibit quasi-periodic cycles.

Finally, the manifest similarity between certain human organs and fractals should be pointed out. The bronchial tubes are just one example of this. Why is their structure almost fractal in nature, endowed with branchings and more branchings? Possibly because the fraction dimensions of fractals makes them perfect for passing between dimensions. Hence, bronchial tubes, with fractal dimensions of approximately two and a bit are the perfect bridge for passing from a three-dimensional transportation tube (dimension 3) to a two-dimensional plane of diffusion (dimension 2); in other words, for passing the oxygen in the air to the blood.



If the cerebral dynamic is wrinkled, defining a sort of strange attractor (left), the person is healthy. However, when it becomes periodic, defining a limit cycle (right), this means the patient is suffering from an epileptic fit (source: C.J. Stam, "Nonlinear Dynamic Analysis of EEG and MEG: Review of an Emerging Field", Clinical Neurophysiology *116/10, 2005).*

QUANTUM CHAOS: THE FINAL FRONTIER

Could the indeterminism of subatomic particles be the fruit of the unpredictability we associate with chaos? No. Chaos theory requires non-linear equations, whereas quantum mechanics is based on a single linear equation (Schrödinger's wave function). Consequently, we can be sure that there is no quantum butterfly effect, since the linearity of quantum equations is not consistent with the non-linearity required by chaos. However, in going from a classic chaotic system to the associated quantum system (by getting smaller and smaller), although chaos disappears, it leaves traces in the form of interlinked fluctuations. The study of these traces has been given the name 'quantum chaology' or 'postmodern quantum mechanics'. And while classical mechanics is deterministic it is also chaotic, but quantum mechanics is probabilistic (determined by chance and probability) but regular. Hence, quantum mechanics has saved us from the curse of chaos… although at the price of making electrons, photons and all other quantum particles appear to be crazy.

A new unpredictability

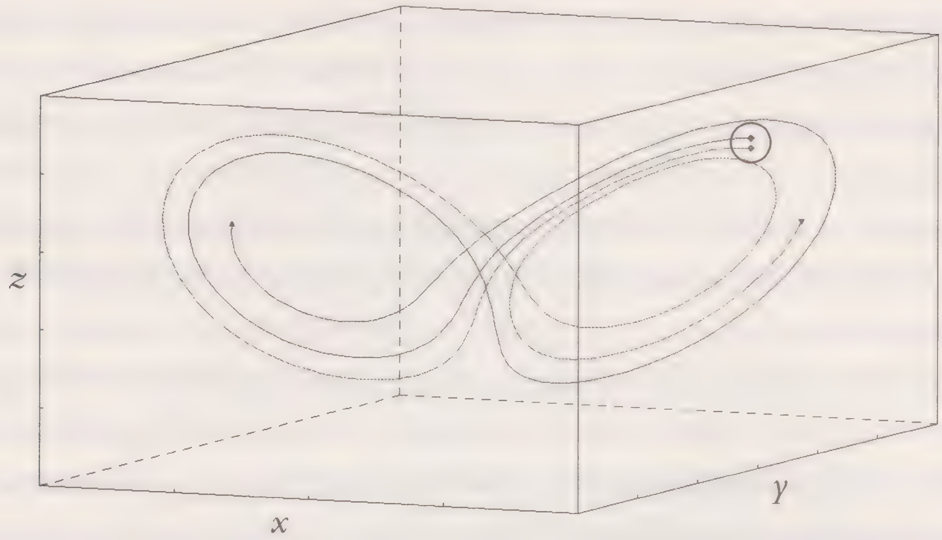
The Marquis de Laplace firmly believed that a deterministic system, a system that follows the laws of Newton, had to be predictable. However, the new discovery has been that a chaotic system that obeys Newton's laws can in fact become chaotic and practically unpredictable. Hence, one of the most revolutionary and dramatic consequences of chaos theory is that it refutes the equation 'determinism = predictability'.

The most likely reason why, over the course of three centuries, determinism and predictability have naïvely been identified with each other, has been the tendency to consistently consider just linear systems and ignore non-linear ones. In this way, the entire Universe seemed like a toy mechanism, as predictable as launching a cannonball or the workings of a clock.

Chaos, as paradoxical as it might seem, is deterministic – it is generated by fixed rules that leave nothing to chance – but imposes fundamental limits on prediction. If we commit a small error in measuring the initial state of our system (this always happens, since in the real world we operate with roundings and truncations), the dynamic equation will generate a prediction that will have propagated and amplified the initial error. Thus, the obstacle to prediction lies in the join between a practical reality (the finite precision of any measurements) and the chaotic mathematical structure of the evolution equation (that inflates the error exponentially). The validity of the prediction is severely restricted, especially in the long term.

However, while chaos is unpredictable, it is still deterministic; in other words, if two practically identical chaotic systems of an appropriate type are driven or powered by the same signal, they will give the same output, although no one can say what output this will be.

Chaos is, let us repeat once more, the random or stochastic behaviour that occurs within deterministic systems. Hence, for example, the classic example of throwing a die can become an example of a random process that is, in reality, perfectly deterministic. The throwing of a die occurs in the same way as launching a cannonball. The only problem is that it is difficult to predict or calculate the particular face of the die that will come up, since any small change in the position and speed at which it is thrown will give rise to a completely different result. The source of the randomness resides in the choice of the initial conditions. Unless we have exact control over them, it is impossible to make a prediction.



Separation of two trajectories that were initially close together in the Lorenz attractor. Both start in the same neighbourhood (indicated by the circle). However after a certain period of time, each ends up in a different lobe of the attractor.

Philosophically, chaos raises an extremely serious issue that even affects the scientific method. The classical procedure for testing a scientific theory consists of making predictions and trying to verify them. However, if phenomena are chaotic, it is impossible to make predictions in the medium to long term – that is intrinsically impossible. Hence, let us imagine that a mathematician uses certain equations that exhibit chaos to describe a physical process, in other words, using equations that capture a dynamic sensitive to the initial conditions, in which random trajectories are mixed with periodic ones. If, in the manner of classical mathematicians, our mathematician is trying to use an initial condition to make a 'prediction' of where the system will come to rest in the long-term, their response must be: "I can only do this if I measure the initial point with infinite precision". However, since this is impossible in practice, the long-term behaviour of the system is completely open. No physicist who comes across equations such as these will dare to work with them. Why? They will obtain completely random results. In fact, the same thing happened to the meteorologist Edward Lorenz and astrophysicist Michel Hénon, whose works were originally overlooked by their scientific colleagues.

Hence, the philosophical crux of the matter is as follows. Since chaos implies a sensitive dependency on initial conditions, this means that the inevitable errors made when establishing the initial conditions will be exponentially inflated, in turn

meaning that practical predictions of a chaotic model must be 'ostensibly' erroneous. Hence, the question that now arises is what type of model can arise if there must be a spectacular and generalised predictive error? How can the model be both chaotic and useful?

The answer is as follows. Chaotic systems can be enormously predictive in a number of ways, although chaos imposes strict limitations on the availability of 'one' sort of prediction.

The first thing that must be clarified, going against conventional wisdom, is that predictions of chaotic systems can be successful in the short term. Of course, no matter how precise our measurement of the initial conditions is, since there is always an error, this will become significantly inflated. It will not be long before the detailed dynamic evolution of the chaotic system becomes unpredictable, although, *nota bene*, there is no reason for this unpredictability to appear immediately. In fact, while this is the case, it is possible to make short-term predictions regarding the trajectory of a chaotic system. However, in the medium to long term the situation is quite different.

And if it is not possible to make medium- or long-term predictions… does this mean the science is useless? No, the qualitative not quantitative aspects remain. Poincaré provides an explanation with his usual clarity:

> "We come across a physicist or engineer who asks us: 'Can you integrate this differential equation for me? I need it within eight days for a construction that must be finished by a certain date.'
> 'This equation,' we reply, 'is not one of the standard integrable forms, and you know very well that there are no more.'
> 'Yes, I know, but why then is it useful to you as a mathematician?'
> Previously, an equation was not regarded as having been solved unless its solution had been expressed with the help of a finite number of known functions. However that is possible for scarcely more than one percent. We can always solve the problem 'qualitatively', or in other words by trying to determine the general shape of the curve represented by the unknown function."

And indeed, chaos reveals functions, shapes and structures where nobody had suspected them. There is order in chaos. Chance has an underlying geometric shape – the stretches and folds of strange attractors. In these cases, when it comes to

verifying a scientific theory, other factors will need to be taken into account, based on geometric verification (qualitative) in contrast to experiments (quantitative). In the next chapters, we shall discuss the highly topical example of climate change, in which meteorologists and climatologists often sacrifice prediction for understanding. Faced with a non-linear problem on a daily basis, they are required to choose between an accurate model for prediction (which is, by definition, impossible to build) or a simplified model to aid understanding. And, of course, science is not just useful for making predictions, not just as a set of effective recipes, but also for explaining the nature of things.

Descartes, for example, with his vortices and linked atoms, explained everything but predicted nothing. Newton, on the other hand, with his laws and gravity, calculated everything but explained nothing. History has sided with Newton, relegating Cartesian constructions to the realm of gratuitous fantasy and the memories of museums, and has for centuries emphasised the predictive aspects. Newton's theory of gravity won out against the Cartesian theory of vortices, condemning it to the attic of metaphysical theories. And in fact, the same thing occurs with mathematical models of chaos theory – albeit with certain nuances – that occurs with Cartesian models: they are inherently qualitative, not designed for action or prediction, but rather for describing and understanding natural phenomena.

If the mathematics and physics of yesteryear occupied themselves with circles and watches, at present they are interested in fractals and clouds…

Chapter 4

The Mathematics of Climate Change

What can be controlled is never completely real, and what is real can never be controlled.
Vladimir Nabokov

In all likelihood, if humanity were to write down a list of the most pressing problems it faces in the third millennium, one of these would be climate change. It is a polyhedral problem, or in other words, one with many faces. One of these faces is scientific, although as we shall see it also has economic and political ones. Both in this chapter and the following one, we shall approach this all-too-real problem from the perspective of mathematics, since the mathematics of chaos plays an extremely important role.

Mathematics and ecology

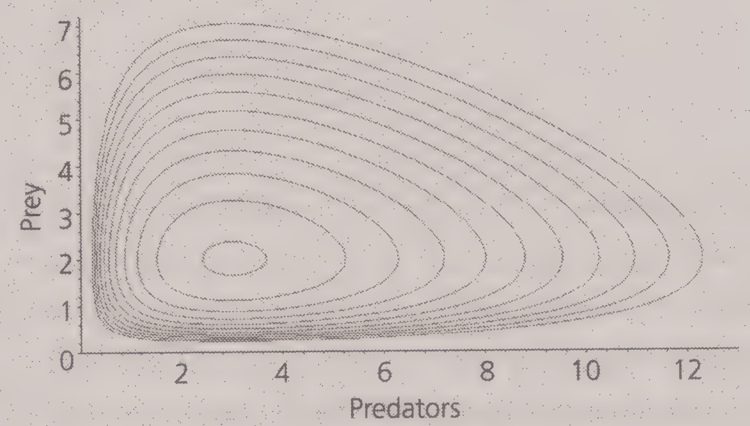
The marriage between ecology and mathematics is nothing new. In fact, mathematical ecology is a branch of biology with a more than respectable history. It came of age some two centuries ago. At that time, numerous scientists began to apply mathematical methods to the study of the relationship between living things and the natural environment.

We have already met one such scientist, Pierre-François Verhulst, who introduced the logistic map for modelling the dynamics of populations. Another was the Italian mathematician and physicist Vito Volterra (1860–1940), famous for formulating a system of non-linear differential equations that described the dynamic of a biological system in which two species interacted, a predator and its prey. However, mathematics have not just been useful in population dynamics, but in the 20th century they have

also been used to model the weather and the climate, two systems in which human beings intervene.

PREDATOR–PREY CYCLE

The predator–prey equations describe, for example, changes in the populations of wolves and rabbits. Rabbits reproduce exponentially, although their number is limited since they are hunted by wolves as food. More rabbits equals more wolves. However, as the number of wolves increases, the number of rabbits decreases, meaning that the wolf population will end up falling. And back to the beginning. The trajectories determined by the system in the phase plane are hence periodic orbits.



Phase plane of the predator–prey dynamic system.

In this context, climate change is a multidisciplinary scientific problem, since it involves climatologists, meteorologists, physicists, geologists, biologists, economists... The cause of this multidisciplinarity lies in the fact that the climate system is complex and made up of five subsystems: the atmosphere (air), the hydrosphere (water), the lithosphere (earth), the cryosphere (ice) and the biosphere (living things). It is impossible to comprehend the infinite complexity of the environment without exploring the multiple links between these systems.

Climate and weather

One of the reasons why climate change has become a hot topic is sociological. Any extreme weather event outside the norm, even though it may have nothing to do with climate change, can be recorded as amateur video footage and subsequently

broadcast by television channels throughout the world. Hence, for example, some in the media attributed the damage caused in Indonesia by the 2004 tsunami and Hurricane Katrina that struck New Orleans in 2005 to climate change. However, the cause of these catastrophes was not climate change, since the first was of seismic origin, and the second was due in part to the poor state of the city's levies.

Perhaps the first thing that we need to do is learn to distinguish between weather and the climate, or in other words between meteorology and climatology. The difference lies in the different timescale to which each refers. The weather is the state of the atmosphere in a specific place at a specific time. For example, today in my city the weather is sunny. However, in contrast, the climate is the state of the atmosphere that has been observed over the course of years. In other words, the climate is, by definition, the averaged state of the atmosphere observed as meteorological weather over more than 30 years. For example, my city has a wet climate, since, generally speaking, the weather is rainy. Hence, on a specific day of the year, Madrid, Antigua and Edinburgh may have the same weather, despite being located in areas with different climates.

Hence, the climate is the periodic series of weather in a given place, which determines the most frequent state, or in other words the 'least anomalous' state of the atmosphere. Consequently, extraordinary events, such as a tsunami or hurricane (except when repeated regularly) are, in principle, in no way related to the climate or climate change.

Global warming

But what is climate change? Does it exist? In fact, strictly speaking, the climate has changed, is changing and will change. As we shall come to mention further on, that is because it is a dynamic system, or in other words, a system that evolves with time. For example, at the end of the 10th century, when Erik the Red led the Vikings to Greenland, he discovered a land covered by grass and without ice (hence a *green land*) which led him to found a prosperous colony. However, later, at the start of the 15th century, when the Little Ice Age occurred, the glaciers advanced and wiped out the Viking colonies. However, this has not been the only fluctuation of the climate in our era. If we look back to the 1st century AD, we find a period of warming that coincided with the fall of the Roman Empire, or, without needing to go so far back, from the second half of the 19th century, after the end of the Little Ice Age, we enter

another period of warming in which we currently find ourselves, only interrupted by a slight cooling between 1940 and 1975.

It is often surprising how short-lived ideas can be. In the second half of the 1960s, which coincided with this period of global cooling, many ecologists were talking of imminent glaciation. In fact, many researchers stated that human activity, by increasing the level of carbon dioxide (CO_2) present in the atmosphere, was causing a marked cooling of the climate. However, at the start of the 21st century, our fears have performed an about turn. There is a consensus that we are in the midst of a period of global warming, the main cause of which is man-made.

At present, the theory of climate change consists of the conjunction of three hypotheses that are not always differentiated, in spite of the fact that each has a different degree of corroboration. These are the three pillars of the consensus.

1. There is a *global warming* of the Earth.

2. The main cause of the global warming is the *greenhouse effect.*

3. The main cause of the greenhouse effect is CO_2 emissions of *human origin.*

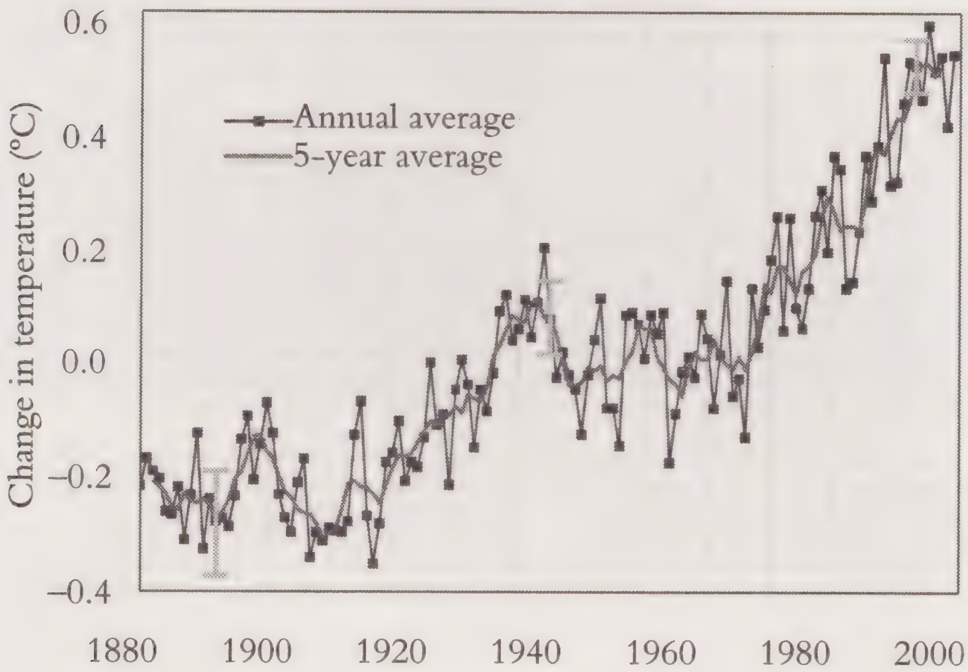
Put another way, climate change = global warming + greenhouse effect + human origin.

The past and present of the Earth's climate

In its fourth assessment report, published in 2007, the UN Intergovernmental Panel on Climate Change (IPCC) estimates global warming during the 20th century at 0.74 °C, and notes that it is more pronounced in the northern hemisphere than in the southern hemisphere.

It is interesting to consider the details of the following graph, which shows the change in the global temperature up to the present day. In first place, we can see that the rate of the Earth's warming since the end of the Little Ice Age, which occurred around 1880, has not been constant. In 1940 and 1975, the warming stalled completely and gave rise to a slight cooling, which caused that spread of panic at the prospect of new glaciation. However, since the 1980s the rate of warming has accelerated markedly. Nonethelesss, between 1999 and 2009 the global temperature

practically stopped increasing, in spite of the fact that nine of the 10 hottest years on record have occurred during this interval.



Increase of global temperature between 1880 and 2010, according to NASA's Goddard Institute for Spaces Studies (GISS).

In fact, what has happened throughout the 20th century is extremely interesting, as the physicist Manuel Toharia explains in his book *El clima: el calentamiento global y el futuro del planeta* (*The Climate: Global Warming and the Future of the Planet*):

> "At the height of industrial development, with an enormous increase in the burning of coal and hydrocarbons over previous decades, the warming stops and a general fall in temperature begins at the start of the 1940s. In the 1970s, during this period of relative cooling, the most widely accepted theory was that of global cooling, which appeared to be leading us to a new glaciation. The idea was that atmospheric pollution caused by industry, car exhaust and heating boilers was making the air more opaque, making it harder for the Sun's radiation to reach the Earth. Nuclear winter models were in vogue."

Yet, since 1980, the global temperature has once again begun to rise sharply. The truth is that the average global temperature is always changing as a result

NUCLEAR WINTER AND THE COLD WAR

The quiet conflict between the Soviet Union and the United States in the midst of the nuclear era, the so-called 'Cold War', saw the development of a theory of an extreme climate phenomenon: as a consequence of a hypothetical atomic war between the superpowers, a drastic global cooling effect could be caused by all the smoke produced. Theoretically, this stratospheric smoke would partially hide the Sun, causing the collapse of agriculture, ensuing starvation, and perhaps even triggering an artificial glaciation. In fact, fear of this nuclear winter was a determining factor in the initiation and signing of nuclear disarmament treaties between the superpowers.

of various factors, such as the eruption of the Pinatubo volcano in 1991, which reduced the average temperature by a number of decimal points, whereas the intense El Niño phenomenon in 1998 made it one of the warmest years of the century.

Accepting that variability is one of the essential properties of the global temperature, and the Earth's climate in general, climatologists attempted to explain the cause of the recent period of global warming by casting their gaze back to the last millennium, since if we can understand what is abnormal in the warming of the Earth by almost one degree centigrade during the last century, perhaps we can discover and attack the causes. A textbook case of scientific method.

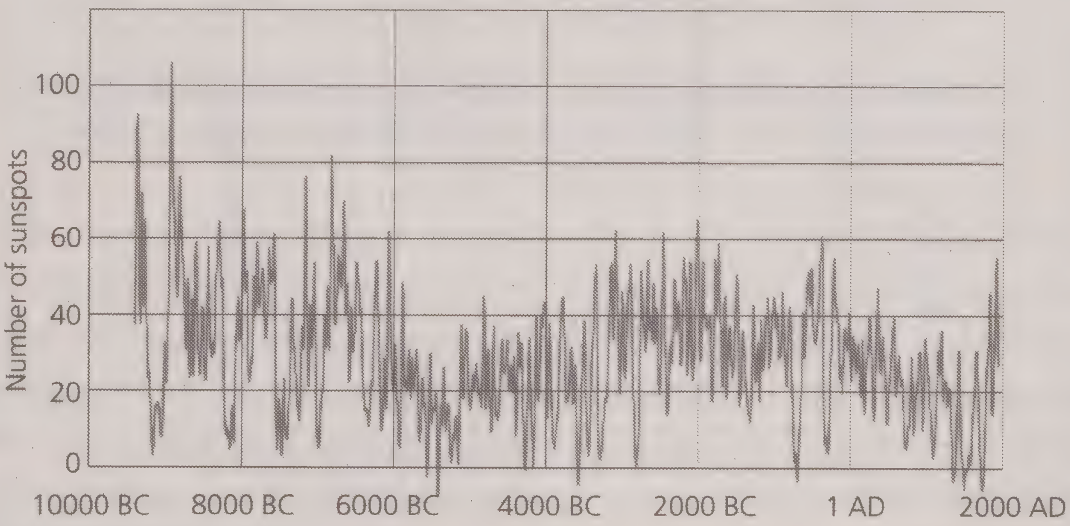
Hence, there is no need to limit ourselves to the study of the Earth's climate at present, but it is also necessary to accurately analyse the past, or in other words, study the history of the Earth's climate. During the last millennium, for example, the actual period of warming was preceded by a Little Ice Age, a result of minimal solar activity and a high level of volcanic activity, which lasted from the 15th century until well into the 19th, and caused the end of the Mediaeval Warm Period, or the Mediaeval Climate Optimum – an extraordinarily hot period that coincided with a solar maximum (the Sun was covered in spots).

Furthermore, at certain other points in the geological history of the planet, the average temperature of the Earth has tended to vary in a proportion similar to that at present, according to evidence from studies in various branches of palaeoclimatology. However, we do not yet fully understand the causes of this warming and cooling of the climate in the past, since there are too many factors involved: the Sun and its cy-

cles, volcanic eruptions, the currents of the oceans, the greenhouse effect, etc. And all this represents a measure of the inadequacy of our current knowledge.

SUNSPOTS

First discovered in the 17th century by Galileo and the Jesuit Christopher Scheiner, thanks to the telescope, sunspots are regions of the Sun whose temperature is lower, but with intense magnetic activity. The darkness of the sunspot is caused by the contrast with its brighter, hotter surrounding area. However, although it may seem paradoxical, the Sun is more active when there are more sunspots, since the areas surrounding the spots are much brighter.

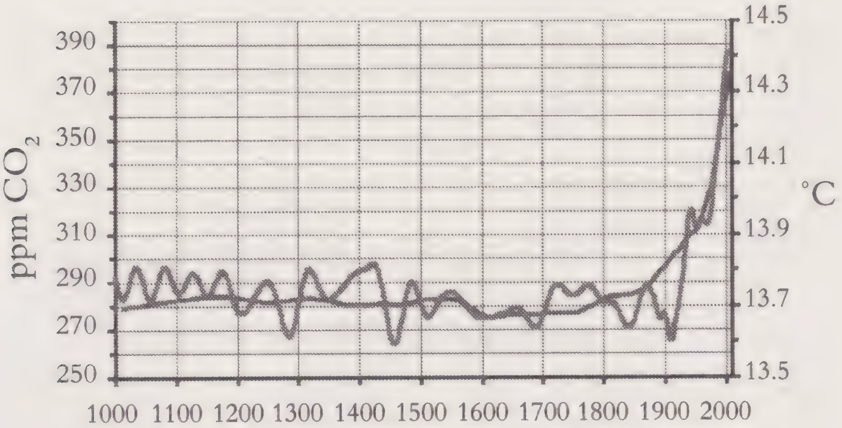


Reconstruction of the number of sunspots during the last 11,000 years.

Furthermore, the recent nature of instrumental meteorological series (temperature recordings at weather stations only go back to 1850) requires the use of proxy-data to analyse climate trends. Proxy-data is climate data indirectly extracted from other sources, such as by dating isotopes of strata of lake sediments, the analysis of fossilised air trapped in bubbles of ice cores or studying the rings of trees.

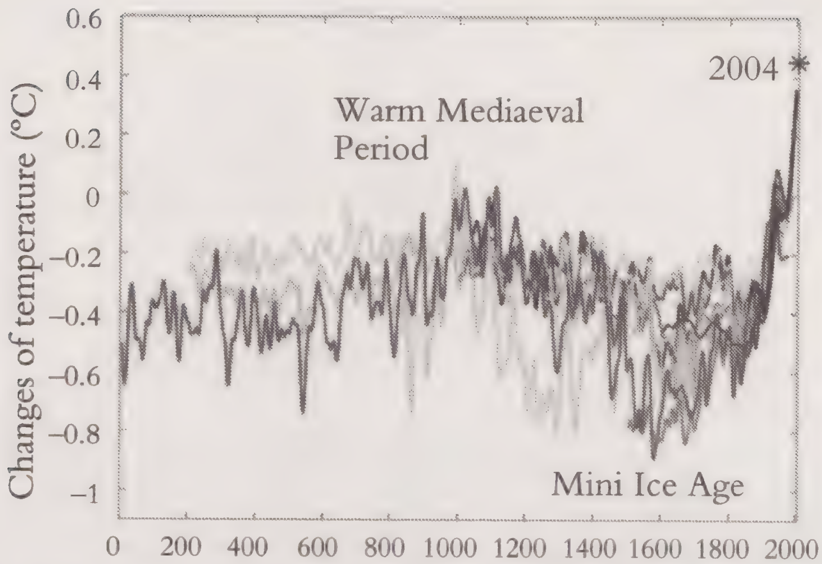
The problem is that reconstructing the temperatures of the past using proxy-data is not guaranteed to be reliable, as shown by the famous controversy over Michael E. Mann's 'hockey stick' that went all the way up to the Congress of the United States.

Although there are other hockey sticks, Mann's is the most prominent and was used as a paradigm by the UN and the IPCC in 2001 (it had been corrected in the last report for 2007). In 1998, the climatologist Michael E. Mann published a famous graph in the prestigious scientific journal *Nature*.The graph was shaped like a hockey stick and a supposedly represented the evolution of the global temperature in the last millennium by studying the annular rings of a certain species of pine. In it, both the Mediaeval Warm Period and the Little Ice Age had practically disappeared and only the temperature peak of the 20th century stood out. It was as if during the last millennium, nothing of importance had taken place until the 'big warming' of the 20th century.



A diseased graph: Mann's 'hockey stick'.

However, various groups of researchers published important critical studies. For example, two of these, the mathematician Stephen McIntyre and the economist Ross McKitrick, reviewed Mann's work and found multiple errors, both in gathering the data and carrying out the mathematical calculations. In fact, McIntyre and McKitrick obtained completely different results using identical data. And they discovered that Mann and his colleagues had used a surprising formula which, regardless of the *input* data, the *output* data always described a graph shaped like a hockey stick. In other words, Mann's statistical analysis of the data underestimates multiple temperature fluctuations. In contrast, the corrected versions of Mann's calculations smoothed the difference between the actual global temperature and that of the Mediaeval Climate Optimum.



Different reconstructions of the global temperature in the last two millennia (note the peaks corresponding to the Mediaeval Warm Period, the Little Ice Age and, especially, the present).

What do we know for sure about the correlation between temperature and CO_2? In 1896, the Swedish scientist Svante August Arrhenius published an article entitled *On the Influence of Carbonic Acid in the Air upon the Temperature of the Ground*, highlighting a direct relationship between the variation of the temperature and the level of CO_2 that would later become known as the 'greenhouse effect', although the first person to use the greenhouse analogy was the mathematician Jean-Baptiste Joseph Fourier, in 1824.

Carbonic anhydride or carbon dioxide (CO_2) is referred to as a 'greenhouse' gas because it retains part of the energy emitted by the Earth's surface – as a result of having been heated by the Sun. That recreates what happens in a standard greenhouse, with glass panels that do not allow the light that penetrates to escape as heat. The result is that in both cases, significant warming occurs. However, CO_2 is not the main greenhouse gas. This is actually water vapour, or steam, responsible for 60% of the effect. Here it is necessary to distinguish between the 'natural' greenhouse effect that makes the Earth inhabitable, increasing the temperature of the planet by 33 °C (from −18 °C) and making life possible, and the 'artificial' greenhouse effect caused by humans as a result of the industrial emission of CO_2, methane, nitrogen oxides and other gases. It is a proven fact that since the Industrial Revolution, humans have polluted the atmosphere, altering its chemical composition, above all as the consequence of the consumption of fossil fuels (coal, oil, etc.).

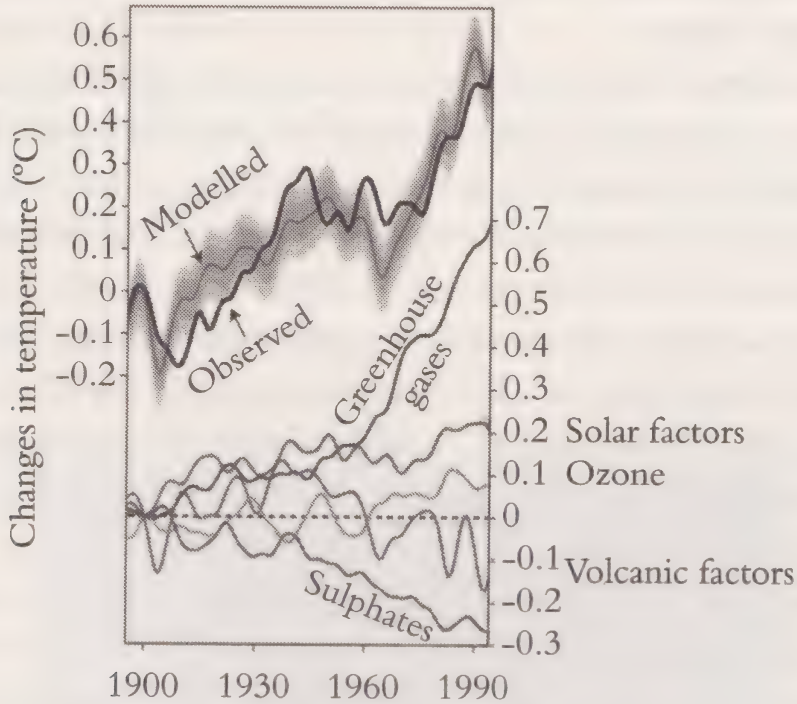
Battersea power station in London, a former coal-fired plant now world famous for appearing on the cover of Pink Floyd's Animals *album.*

However, the dynamic of the climate is much more complex than simply saying that the level of CO_2 increases and the temperature rises. In fact, it is hard for the natural and human variability of CO_2 to explain the increase in temperatures between 1920 and 1940, when the levels were much lower, and harder still the cooling that occurred between 1940 and 1975, when there was considerable growth in emissions of human origin. In fact, palaeoclimatic studies show that temperature does not strictly follow levels of CO_2. In certain reconstructions temperature peaks appear some 800 years before peaks in the concentration of CO_2.

Hence, greenhouse gases aside, other factors lie behind the rise in temperatures, both natural, such as solar activity, and human, such as the heat caused by the urbanisation of the continents or changes in land usage, two examples of human processes that have been habitually underestimated, but which nonetheless could lie behind some of the observed warming. Similarly, just as there are natural factors that tend to cool the planet, such as volcanic activity, there are also human factors that favour this. One of these is the so-called 'global dimming' phenomenon, which refers to the gradual reduction of the quantity of sunlight that has reached the surface of the Earth since the 1950s. It is believed that this dimming has been caused by an

increase in particles such as charcoal and sulphates in the atmosphere from human activities, mainly combustion and air transport.

To summarise, mathematical models and graphs devised by climatologists are designed to evaluate these factors together, as well as their 'impact' or contribution, both positive and negative, to the evolution of the global temperature. When the climate models only take into account natural factors, they do not reproduce the increase observed in the average global temperature. In contrast, when human factors are taken into account, it is possible to reproduce the warming of the last 35 years. Hence, the scientific community tends to believe that while volcanic eruptions and sulphates lower the temperature of the planet, greenhouse gases and solar activity contribute to its increase. And the sum or balance of all these causes explains the increase of the global temperature of the planet in the 20th century by 0.74 °C.



Variation of the global temperature according to different positive and negative factors.

Returning to statistics and chaos theory

However, in all these graphs and the mathematical models on which they are based, there are a number of uncertainties that scientists attempt to quantify. When it

GLOBAL DIMMING AND 9/11

During the almost complete grounding of air traffic for the three days after the 9/11 attacks in 2001, two US scientists, David Travis and Gerry Stanhill, took advantage of the situation to measure the variations in the temperature in certain parts of the United States. The results were incredible: both observed a variation of 1°C in the daytime temperature. In other words, after three days without flights, the temperature had dropped by almost one degree.

comes to evaluating the uncertainty of these models, which attempt to reproduce the Earth's climate in the past, statistics has much to say, as we shall see further on, whereas when it comes to quantifying the uncertainty of the models that attempt to predict the climate in the future, chaos theory comes to the fore, as we shall see in the following chapter.

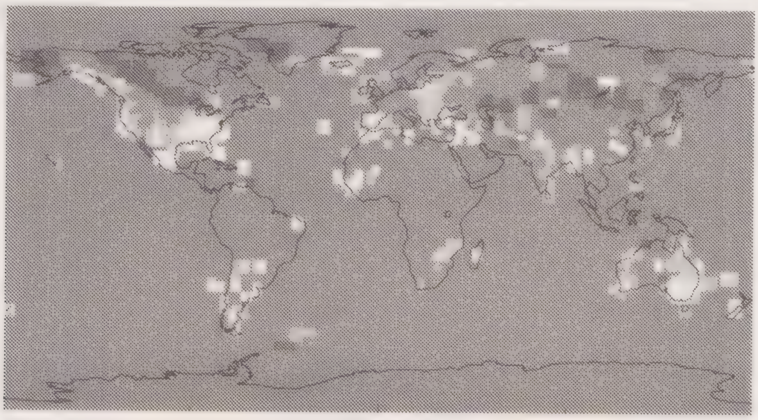
Until now, we have continuously been discussing the global temperature of the planet, whether it increases or decreases, how it varies... But what exactly is the 'global temperature'. Has anyone asked? While the meteorological weather and the local temperature at a certain location can be observed and measured, the climate and global temperature of the planet are not measured but are the results of a previous calculation, an estimation. We do not have a giant thermometer that we can place on the Earth to determine the exact temperature. Once way of considering the global temperature is as the result of a 'statistical recipe'. It is an average that can be calculated in many different ways based on the data provided by weather stations, weather balloons and satellites.

This is illustrated by the mathematician Christopher Essex and the economist Ross McKitrick. Imagine that a physics teacher is teaching his or her students how to calculate the average temperature of the classroom. In winter they have taken four temperatures in different positions (beside the door, beside the window, at the teacher's desk and from the seats at the back), which give measurements of 17°C, 19.9°C, 20.3°C and 22.6°C, respectively. In spring, the teacher opens the window and allows a warm breeze to enter the room. At that point, the four thermometers register 20°C and the teacher asks the students whether the room has become warmer or colder with respect to winter? Half the students calculate the average of the winter temperatures using the arithmetic mean, or rather by adding them

together and dividing by four. The other half decides to calculate a sort of quadratic mean, adding the squares of the temperatures, dividing by four and taking the square root. What conclusions do the groups draw?

The group that used the first method, the linear method, obtained a winter temperature of 19.95°C. Hence, it increases by 0.05°C to 20°C in spring. In contrast, however, the group that used the second, quadratic method obtains a winter temperature of 20.05°C and comparing the two temperatures, the classroom has cooled by 0.05°C in the spring. Which group is right? Both, since the two methods used to calculate the average of the mean winter temperature are valid (the only difference is how they reach the thermodynamic equilibrium).

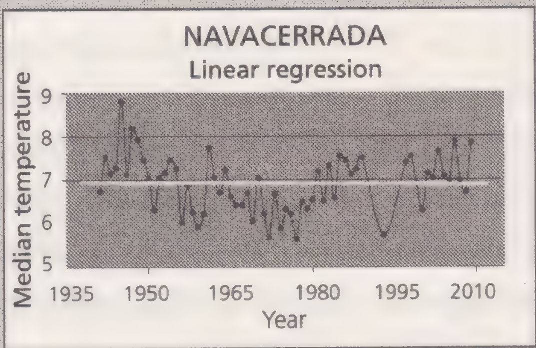
Furthermore, when we go from the classroom to the planet, there is a problem of quantity and quality with respect to the initial data, since there is no network of weather stations that is properly distributed in space and time (the use of weather balloons became widespread from the 1950s, and the use of climate satellites from the 1980s). Hence, throughout the whole world, just 1,000 stations have records that cover the whole of the 20th century and most of these are located on the ground in the northern hemisphere (in European and North American cities mainly), meaning that the oceans and the southern hemisphere are completely relegated to the background. Hence, given that the network of observatories used to calculate the variation in the global temperature over the past century is small and poorly distributed, the extrapolation will be subject to significant error.

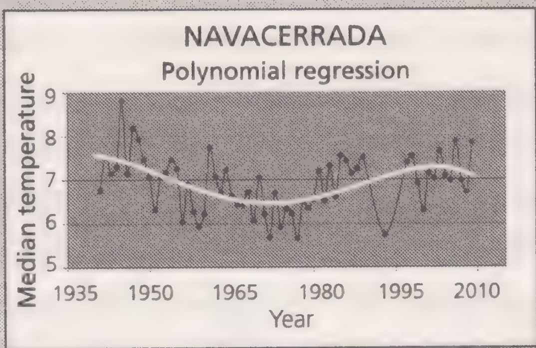


Network of observatories with records spanning the period between 1880 and 2009. Note that the majority of the planet is not covered by a cell, since there are no observations in those positions (source: GISS simulation).

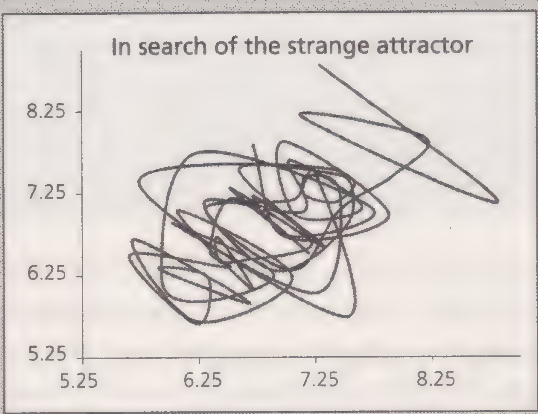
A STATION AT 1,888 METRES

If we focus our attention on the Iberian Peninsula, we will see that for the calculation of its global temperature, both the UN IPCC and NASA's GISS only take into account around 20 stations, only four of which are located in rural areas, sufficiently far from the major cities, and of these four, the Navacerrada weather station north-west of Madrid is the only one located high up in the mountains with a sufficiently broad historical record. Now, if we study the temperature curve in the same way as climatologists, using a linear regression, the first thing we see is that the overall trend (linear) of the temperature at Navacerrada throughout the course of the 20th century has been to remain constant. However if, in order to increase the adjustment, we carry out polynomial regression – in other words, instead of fitting to a straight line, we look for a soft curve that provides a good fit for the series of temperatures – we obtain two trends that are separate in time, downwards and upwards:





And if, making use of the method devised by the mathematicians of chaos, David Ruelle and Floris Takens – which consists of, given a series in time, a, b, c, d... drawing the associated path (a,b), (b,c), (c,d)... – we attempt to find the dynamic (chaotic?) and the attractor (strange?) for the climate at Navacerrada, we get a curve whose complex trajectory can be interpreted as a sign of chaos:



What's more, many of these stations are located in cities and hence are subject to what is referred to as the 'heat island' effect, which modifies the readings (the tarmac, cars, street lamps... significantly skew the temperature readings for cities with respect to their surroundings), and each calculation centre corrects or 'normalises' these anomalies in their own way.

To summarise, we now know much more about the climate than we did before. At present there is widespread concern about climate change grounded in established facts and forecasts based on observed trends and the application of models. These facts and provisions are reasonable, but they raise a number of uncertainties. We now know from the models that it is not possible to explain the global warming that has been observed solely in terms of natural factors, and it is likely that greenhouse gases – mainly, but not exclusively, caused by the human use of fossil fuels – are the largest cause of this warming, together with changes in land usage, including agriculture and deforestation.

However, all this does not detract from the fact that, at the same time, we are conscious of the errors that may lie behind them. The models may be overly simple and imperfect, the measurements of the data may be too inaccurate, imprecise, etc. Hence, there are various sources of uncertainty in climatological predictions, such as the scarcity of data on certain variables and for certain areas of the world, or the limitations of our understanding of the operation of certain mechanisms (the effect of aerosols and dust particles). Furthermore, the geological records show that in past centuries and millennia there were significant fluctuations in the climate that cannot be attributed to human influence and must be taken into account when analysing the current climate.

Chapter 5

Chaos, Weather and Climate

Prediction is difficult, especially of the future.
Niels Bohr

"Jack Hall is a climatologist who warns that global warming could trigger an abrupt change in the Earth's climate. And his predictions are based on the melting of the polar ice caps causing large quantities of fresh water to enter the oceans. This results in the interruption of the Gulf Stream in the Atlantic, destabilising the climate in the northern hemisphere. Furthermore, shortly after, a series of inexplicable events begins to occur – snow falls over New Delhi, enormous chunks of ice rain down over Tokyo, a powerful tornado destroys the buildings of Los Angeles, and Manhattan is buried under a giant tsunami which then freezes. The result is a global mega-storm that creates a new ice age and then buries the planet below tens of metres of ice."

Broadly speaking, this is the plot from the film *The Day After Tomorrow*, filmed in 2004 and highly successful at the box office. However, its plot is closer to science fiction than real science, despite the fact that the future predicted by the latter is also far from promising. Just as in the previous chapter we cast our minds back to observe the Earth's climate in the past, let us now look into the future. What do we know about the future climate? Can it be predicted? Do mathematical models of the climate make it possible to predict, for example, the average temperature of the planet over 100 years?

The future of the climate: an impossible prediction

Mathematical attempts to model the weather and the climate began in the Roaring Twenties. At that time, synoptic weather forecasters (who base their forecasts on observations) realised that, if they wished to extend the range of their predictions

of the weather or, in general, the climate, they would require the indispensable help of dynamic weather forecasters (who make regular use of equations). Shortly after, the atmosphere, just like the ocean, was found to be a highly complex and dynamic system. The pioneering idea proposed at the start of the 20th century by the Norwegian physicist and meteorologist Vilhelm Bjerknes (1862–1951) of solving the problem of weather and climate forecasting by resolving the equations that describe the atmosphere would be no mean feat.

The English mathematician Lewis Fry Richardson (1881–1953) returned to Bjerknes' dream later on. Taking advantage of his travels throughout France as an ambulance driver during World War II, he gathered a broad set of meteorological data for a specific day, 20 May 1910, and for six weeks carried out thousands of additions, subtractions, multiplications and divisions in order to provide a six-hour forecast for a small region. Unfortunately, the result was completely unsatisfactory. His forecast did not match the data that he had gathered. However, far from losing hope as a result of the failure, he had a premonition: "64,000 people will be needed working in shifts to predict the state of the atmosphere more quickly than its real evolution." Decades later, his 'weather machine' became transformed into reality, albeit in an unimaginable way. Instead of 64,000 people, there were 64,000 electronic valves.

The global meteorological weather and climate system had to be represented by a system of equations with more than 5 million variables, mixing three types of ingredients. First, the fundamental principles of physics (the conservation of energy, mass, etc.); second, the appropriate mathematical equations (the intractable non-linear Navier–Stokes equations for the motion of fluids); and finally, certain empirically induced formulae (such as the formula for the evaporation of water as a function of humidity and wind speed).

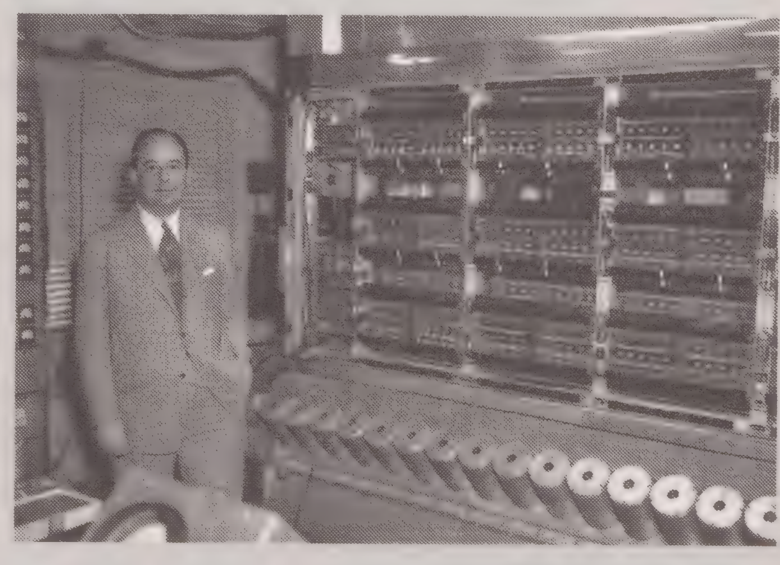
However, the lack of effective computing tools delayed its development until the rise of computers, in the second half of the 20th century. Finally, the study of a system as complex as the atmosphere came to be feasible thanks to a fertile methodology comprised of mathematical modelling, analysis and simulation using computers that made possible corrective actions designed to improve the situation, or in other words 'control'.

The distinction between meteorology and climatology is based, as we know, on the different time scale to which each makes reference. While weather forecasting deals with a couple of days, or at most, one to two weeks, climate forecasting, on

THE VON NEUMANN PROPHECY

John von Neumann (1903–1957) was an exceptional mathematician who showed great skill in almost all areas of mathematics: set theory, functional analysis, quantum mechanics, economics… Involved in the development of computers, von Neumann set himself the challenge of being able to predict the weather and the climate using computers. In this respect, in 1955 he wrote that "intervention in atmospheric and climatic matters will probably come in a few decades, and will do so on a scale that is hard to imagine at present". Among the members of the group with whom he worked at Princeton was Jule Gregory Charney (1917–1981), a highly influential meteorologist and climatologist who came to lead much of the cutting-edge research and … was Edward Lorenz's 'boss'.

On 31 January 1949 the powerful ENIAC computer, with Von Neumann and his colleagues at the helm, was able to provide a 24-hour forecast for the development of a large storm over the north-west of the United States. A major milestone in the history of meteorology.



John von Neumann with the ENIAC computer.

the other hand, takes place on a horizon that may span up to several centuries. Furthermore, while the first discipline attempts to obtain high levels of accuracy in its predictions, the second is more qualitative than quantitative, since it only seeks to understand the climate, which is, by definition, the average state of the atmosphere that can be observed as meteorological weather over various years.

In meteorological studies, the unknown is the specific and instantaneous temperature, or rather the temperature of a given place at a given moment in time – for example, the temperature at the Brandenburg Gate (Berlin) tomorrow at 12:00. On the other hand, in climate studies, the unknown is the average temperature – for example the average temperature in the city of Berlin for the year 2100. This average, which is spatial and temporal, is calculated by averaging (integrating) the temperature of each point of the city for every day of the year.

Another issue is how it is possible to do this in practice since, generally speaking, only the temperatures for a discrete quantity of points and times are known (such as weather stations), which are then used to interpolate and calculate the average temperature. However, the interpolation and the average have no reason to be the same, as noted in the previous chapter (think of the experiment of the physics teacher and their students).

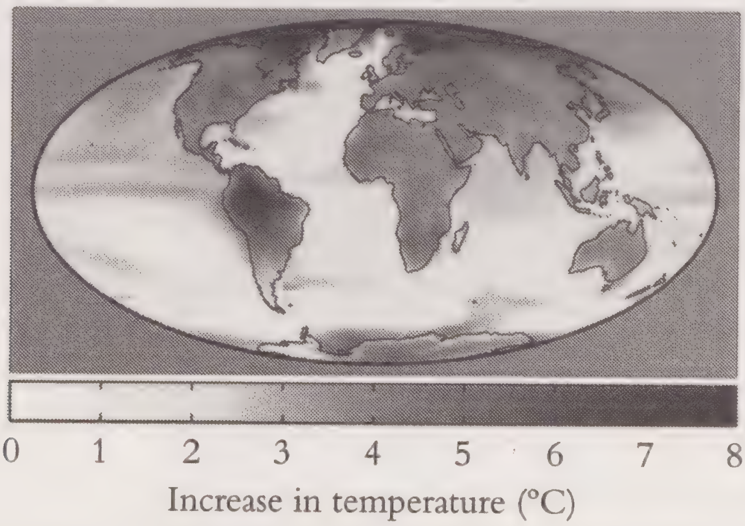
Hence, the tools available to meteorologists and climatologists for studying the evolution of both types of temperature are models based on the equations of the motion of a compressible and stratified fluid (the atmosphere) over a bumpy sphere in motion (the Earth). And clearly the model depends on the 'initial conditions' and the 'environmental conditions'. The initial conditions – for example, the temperatures today – are of greater importance in weather forecasting, whereas those related to the environment – such as the different behaviours of the air depending on whether it is in contact with the ocean or the Earth – are more relevant to climate forecasting.

One type of climate model that is extremely useful for diagnostics is 'energy radiation balance models', whose grandfather was the Swede, Svante Arrhenius from the end of the 19th century, and whose fathers are M.I. Budyko and W.D. Sellers (1969). The fundamental basis of these models is a differential equation that equates the derivative rate of change of the average temperature with respect to time to the addition and subtraction of various factors (such as the solar radiation absorbed by the Earth, which is subtracted from the radiation emitted by the Earth as a hot body). According to how these factors are weighted in the function of the average temperature, the energy balance model will be closed in one way or another.

This type of model can become complex and achieve the high degree of sophistication that characterises current 'general circulation models', which cover the totality of the Earth's surface. However, as is logical, given their extreme complexity, there is no known analytical solution, and they can only be tackled numerically.

Naturally, the numeric solution is also extremely difficult, since it requires an incredible amount of computer-assisted calculation and has the drawback that, since it requires considerable computing time, it is not possible to use a fine grid (an area such as the British Isles, for example, is covered by little more than a dozen points), with the uncomfortable artificiality that this entails.

In the international scientific community there are a number of groups that have developed general circulation models, since these are the models used by the main climate prediction agencies (such as the UN IPCC and NASA's GISS). Each of these perfects the methods for modelling the physical processes and the numeric resolution techniques for the equations in line with advances in observational knowledge and the simulating power of computers.



Prediction of the rise in average temperatures between 2070 and 2100 according to the IPCC HadCM3 general circulation model.

Certainty and uncertainty in mathematical models

However, let us backtrack and return to the 1960s, when a young meteorological colleague of Jule Gregory Charney, Edward Lorenz, whom we have already come across, suggested the 'curious' model made up of three normal differential equations to describe the motion of the atmosphere, now known as the 'Lorenz system'. As we saw in Chapter 2, Lorenz discovered a chaotic behaviour in the solutions, which made his system, in practice, unpredictable. Any error in the observation of the current state – and in a real system, this is inevitable – may render an acceptable prediction of the state in the distant future impossible. In his own words:

> "When my results are applied to the atmosphere, which is ostensibly non-periodic, they indicate that the prediction of a sufficiently distant future is impossible using any method, at least unless the initial conditions are exactly known. In light of the inevitable imprecision and lack of completeness of meteorological observations, long-term prediction appears to be impossible."

Let's go even further back in time, to 1908. At that point, Henri Poincaré had already embarked upon in-depth study of the roots of this type of phenomena, that are so unstable and in which it is impossible to make long-term predictions regarding the dynamic of the system. For his observations, the Frenchman used the three-body problem as the basis of his research, but also – take note – meteorology. In terms of the latter, Poincaré stated that weather was unstable, that meteorologists did not know why and, as a result of this, they were unable to say where or when a storm would occur:

> "Why do meteorologists encounter so much difficulty when it comes to forecasting the weather? Why do rains and storms appear to occur randomly, in such a way that many people who would find it ridiculous to pray for a solar eclipse consider it extremely natural to pray for rain or sunshine? Meteorologists are aware that a cyclone will occur somewhere, but they cannot predict where. An extra tenth of a degree in one point and not another, and the cyclone is triggered here and not there, extending its fury over countries that would otherwise have escaped."

As David Ruelle notes: "The mathematics of Poincaré have played their role, but his ideas on meteorological predictions had to be discovered independently." In an article published in 1963, Lorenz echoed Poincaré's work on dynamic systems but was still unaware of his ideas on chaos, the weather and the climate.

Given that even a model as simple as Lorenz's is chaotic and, furthermore, given that we discovered that chaotic regimes are commonplace among non-linear systems, it is reasonable to conjecture that any precise model of the large-scale behaviour of the atmosphere will also exhibit some kind of sensitivity to the initial conditions – conditions that ensure disturbances the size of those produced by the wings of a butterfly can determine the development of tornadoes. And, it goes

TURBULENT SKIES

Chaotic phenomena that have attracted attention due to their connection to the climate dynamic include the transition to turbulence in fluids, studied, as we have mentioned, by David Ruelle and Floris Takens. They proposed strange attractors as the mathematical explanation of turbulence, a mix of spatial and temporal chaos that had already been considered by the Epicurean philosopher Lucretius some 2,000 years ago. The transition of the fluid from laminar behaviour – constant, stable and regular – to unpredictable turbulent behaviour – non-constant, unstable and irregular – will be familiar to anyone who has travelled on an aeroplane and suddenly passed from calm air into a bumpy air pocket.

The smoke of Humphrey Bogart's cigarette first propagates in a laminar motion before becoming turbulent and chaotic.

without saying, vice versa. Since this variation in the initial condition is whether the butterfly flaps its wings or does not, if the first case is able to determine the development of a tornado over Texas, the second can also avoid it. Or rather, it may cause it to occur over Singapore or, perhaps, New York. Or better still, not to occur anywhere.

The motion – or lack thereof – of a simple set of butterfly wings can today produce a minute change in the atmosphere, such that after a certain period of time the behaviour of the atmosphere will diverge considerably from what would have taken place. In one word, it is chaos.

The consequences of chaos for meteorological predictions are well-known by all: it is impossible to predict the meteorological weather beyond ten days. This is why television weather forecasters often make mistakes when they attempt to predict the weather more than one week in advance during the holiday season: micro errors when it comes to specifying the initial conditions of the atmosphere become inflated until they result in macro errors in their predictions.

AN EXCERPT FROM THE NOVEL *JURASSIC PARK* BY MICHAEL CRICHTON

"The origins of chaos theory lie in the attempts to develop computerised meteorological models in the 1970s. The climate is a large and complex system; specifically the Earth's atmosphere when it interacts with the continental masses and the sea, and with the Sun. The behaviour of this large and complex system defies understanding. If I use a cannon to launch a projectile with a certain weight at a certain speed and with a certain angle of inclination, and I then launch a second projectile whose weight, speed and angle are almost identical, what will happen?"

"The two projectiles will land on almost the same spot".

"That's right. This is a linear dynamic".

"Understood".

"But if I have a weather system that starts with a certain temperature and a certain wind speed and a certain humidity, and I then repeat it with almost the same temperature, wind and humidity, the second system will not behave in almost the same manner: it will diverge and will quickly become different from the first; thunderstorms instead of sunshine. This is a non-linear dynamic."

However, what are the consequences for climate predictions? In his book *The Essence of Chaos*, Lorenz states that:

> "Almost all of the global models have been used for predictability experiments, in which two or more solutions resulting from slightly different initial states are examined to detect the presence of the sensitive dependency… Almost

without exception, the models have indicated that minor initial differences become amplified until they are no longer minor."

In fact, in its 2001 report, the UN IPCC noted that:

> "In the research and creation of climate models, we should acknowledge that we are dealing with a chaotic, non-linear system and, as such, long-term predictions of future climate states are impossible."

In its 2007 report, it adds:

> "Since the work of Lorenz (1963), it has been known that even simple models can exhibit a complex dynamic as a consequence of non-linearity. The non-linear dynamic inherent to the climate system appears in climate simulations at any time scale. The models for atmosphere–ocean, climate–biosphere and climate–economy interaction may exhibit a similar dynamic, characterised by their partial unpredictability, bifurcations and the transition to chaos."

It is fundamental in understanding the claims made at various points in time regarding climate change, to realise that neither the weather nor the climate can be modelled in such a way as to be able to make exact predictions of what will happen next week or within 100 years, respectively. The results produced by computerised models are basically scenarios or simulations with a significant probabilistic component that must be evaluated for each case. In fact, prudently, the IPCC prefers to make use of the term 'projection' instead of 'prediction' when referring to the results of scenarios or simulations. Each scenario or simulation of, for example, the average temperature of the planet in 2100 depends on a series of assumptions (quantity of greenhouse gas emissions, solar variation, etc.) and the problem largely lies in determining which of these corresponds to our real world. We still do not know exactly which aspects of the climate can be predicted with certainty in the long term, since unobservable perturbations may lead to dramatically different futures.

However, it is hard to accept the idea of the inherent unpredictability of the weather and climate in the long term as a result of chaos. In the 1970s, many researchers hoped that by adding more and more variables, the system would stabilise and, hence it would become possible to achieve a degree of long-term

predictability for the behaviour of the atmosphere. Hence, Jule Gregory Charney remained optimistic, stating, "There is no reason why numerical methods should not be capable of predicting the life cycle of a single system, it is just that current models have fatal flaws." However, one of these fatal flaws was, and continues to be, irreparable – they are chaotic.

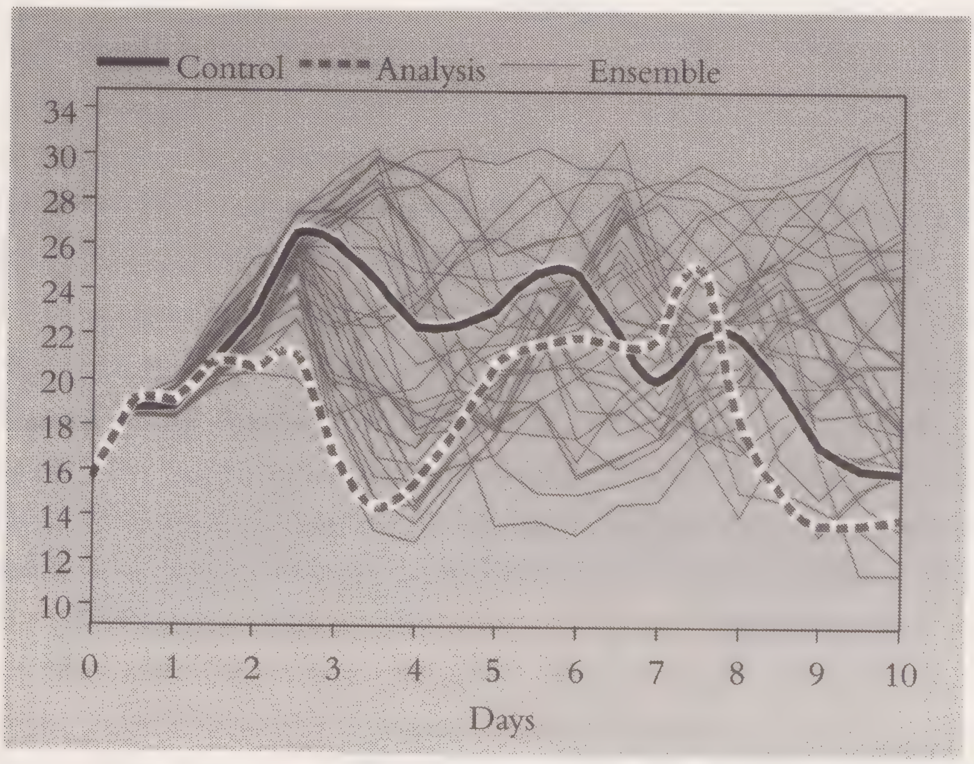
For some scientists, as Tim Palmer (one of the IPCC's leading climatologists) reflects in an article entitled *Global Warming in a Nonlinear Climate – Can We Be Sure?* chaos does not arise to the same extent in climate forecasting as in weather forecasting. In line with a distinction first noted by Lorenz, weather forecasting forms part of an 'initial value problem' (predictions of the first type), in which the butterfly effect exerts a large influence on trajectories. This is why, if we wish to follow the trail of the weather in order to predict it, we must closely follow the trajectory-solution of the equations whose initial conditions are the weather conditions for today (temperature, pressure, humidity...).

In contrast, climate forecasting is part of an 'environment problem' (predictions of the second type), in which the butterfly effect no longer plays such an important role since we are dealing with attractors not trajectories. What is interesting in the study of the climate is the long-term behaviour, which is defined by the attractor. In other words, if we wish to know the climate, it is not necessary to follow one specific trajectory or another, but to observe the long-term behaviour of both, as they approach the attractor, since it provides the average regime of the weather – in other words, the climate. And if, furthermore, we are interested in understanding how the different factors or conditions (atmospheric concentration of CO_2, solar radiation...) influence the climate, we would need to study how these parameters modify the shape of the attractor.

To summarise, if we identify the climate with the attractor of the atmospheric system, the 'butterfly' is squashed. However, since the climate system is non-linear and assumed to be chaotic, the attractor will be strange and probably have a sink of attraction with highly complex fine and rough grain details, whereby it does not advance too much into the domain of instability. For example, if the climate could be identified with the Lorenz system attractor, whereby rotation with respect to the right wing meant rain and rotation with respect to the left wing meant there would be no rain, we would know the general pattern of the climate. On some days it would rain and on others it would not, but little more, since the meteorological trajectories go round each wing of the attractor randomly and in an unpredictable manner.

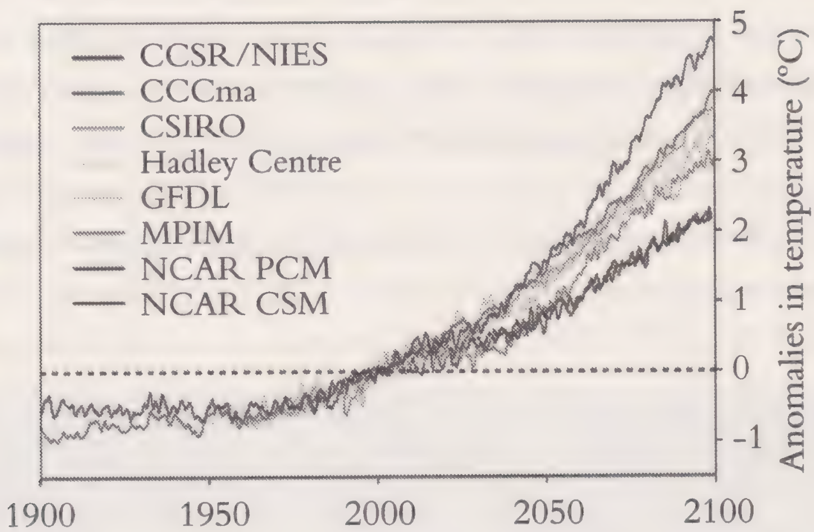
To this day, more than 40 years after Lorenz's discovery, the techniques for short- and long-term prediction have improved considerably thanks to theoretical progress (new understanding), combined with practical advances (faster computers), that have helped minimise the great complexity (chaotic nature) of the weather and the climate. One of these advances is so-called ensemble prediction, which involves using sets of different initial conditions and different mathematical models at the same time. The approach attempts to minimise the errors when it comes to determining the initial condition and the intrinsic errors of the model.

For short-term predictions (meteorology), where the dominant factor is the error in determining the initial conditions, ensemble prediction has been used with proven success for many years with a single model and set of initial conditions. In other words, it studies the evolution of the model for similar initial conditions, before drawing up a forecast comparing the different results. It is often observed that the different outputs (around 50) produced by the model are extremely similar for the first days of the forecast, but after the third and fourth day, growing discrepancies appear in their evolution.



Ensemble prediction of the temperature in London carried out on 26/6/1994 by the European Centre for Medium-Range Weather Forecasts (ECMWF). From the fourth day (horizontal axis) the dispersion covers a strip up to 16°C (between 14–30°C on the vertical axis).

For long-term predictions (climate), where the dominant factor is the error inherent in the model, the ensemble prediction method is used with different models. The evolution is studied based on the same initial condition for various models, and a prediction is then made by weighting the different projections. For example, based on these models, the IPCC predicts that the average global temperature for 2100 will increase by between 2.2 and 4.7 °C with respect to its value for the year 2000. The results of the different computerised models are not identical and the disparity reflects the degree of uncertainty in our knowledge of the Earth's climate.



The global models predict that the average temperature of the planet in 2100 will increase by between 2.2 and 4.7°C; hence there is a degree of uncertainty of 2.5°C.

There is enormous interest in the development of the ensemble prediction technique, since it is envisaged that it will be extremely useful for predicting climate change. At any rate, one thing is certain. It is necessary to radically abandon the idea that we can discover an algorithm that allows us to make universal, long-term predictions of the atmosphere dynamic. Chaos imposes severe limitations on our horizon of predictability.

When mathematics becomes economics

To summarise, Lorenz's achievement was to have proven the chaotic, unstable and unpredictable behaviour of the meteorological weather and, by extension, the climate.

The atmosphere is a non-linear system... and is clearly chaotic. With respect to this latter point, chaos should not be understood as something erratic and disordered, but as a type of order without periodicity. The climate is a chaotic system in the sense that it can suffer unpredictable changes, even in the absence of determining factors. Hence, one of the main goals of cutting-edge research, as has been mentioned, is to find the correct mathematical models for this chaotic climate in order to predict the impossible: our future.

Climatic models are, as we have seen, mathematical models that attempt to simulate the climate of the past and predict that of the future. There is a complex hierarchy of climatic models, running from the simplest, given by a few equations that represent the dynamic of the average global temperature, all the way to the most complex, which require the use of supercomputers and attempt to model the development of multiple climate variables (the average global temperature, wind, humidity, ocean currents...). However, even the most complex models are simplifications, since it has still not been possible to find models able to reproduce the past and provide detailed predictions of the climate on a local, not just global, scale. The shortfalls in computing and predictive power make it difficult to develop models on a detailed scale, required for the analysis of the national and regional impacts of climate change.

Faced with this non-linear problem, scientists must choose between a precise model for prediction (impossible) or an extreme simplification to aid understanding and explanation. In the words of one of the great physicists of the 20th century, Freeman Dyson, "Climate models are essentially tools for understanding the climate, which are still not suitable for its prediction; there is no reason to believe the numbers just because they come from a supercomputer." The shadow of the unpredictability of our chaotic climate requires prudence.

The problem of climate change is that, even when faced with multiple uncertainties, the consequences may be fatal. Although we may not be fully certain regarding future climate change, the risk derived from doing nothing is enormous, since it is a problem that affects us all.

Let us now cover the ground from Montreal to Kyoto. The United Nations Conference on the Human Environment, held in Stockholm in 1972, enshrined the 'principle of precaution' as a guide for environmental policies. In other words, environmental policies had to be regulated at international level to avoid market failures. And the first step taken in this direction was the negotiation and enactment,

in the 1980s, of the international treaty known as the Montreal Protocol to eliminate CFCs (or chlorofluorocarbons) and protect the ozone layer.

With the establishment of the IPCC in 1988 (which has published reports in 1990, 1995, 2001 and 2007), the fight against climate change has also been established at UN level. Later, the Rio Conference in 1992 represented a landmark (not for nothing is it known as the Earth Summit), since it saw the approval of the United Nations Framework Convention on climate change prepared by a group of experts. Five years further on, in 1997, the so-called Kyoto Protocol was agreed, aiming to reduce global emissions of the most significant greenhouse gases by 5.2% over the period 2008–2012, with respect to the emissions for the base year, 1990. The treaty required a modest reduction in emissions of more than 1,000 million tonnes of CO_2 (although as human beings, we emit 2,500 million tonnes a year just by breathing). In 2004, with the signing of the protocol by the Russian Federation, Kyoto came into force, having already been ratified by more than 55 of the 167 member states of the Framework Convention.

The crux of the matter is, as has been mentioned on numerous occasions, that the problem of climate change has various faces, and on top of scientific uncertainties come uncertainties with respect to the costs and benefits of the Kyoto Protocol. Whereas the Montreal Protocol on the use of CFCs generated a rapid consensus, since the final costs were not excessive, the costs of the Kyoto Protocol were excessive for certain countries. Indeed, generally speaking, it is designed along the lines of 'the polluter pays'.

The real problem lies in the fact that, despite the fact that the damage caused by climate change is greater than the costs of the protocol (this is the position of the controversial 2007 Stern Report) commissioned by the United Kingdom, which puts the damage of doing nothing at between 5% and 20% of global GDP), its implementation will only reduce expected global warming by 2100 by 0.18°C, or in other words, by this date, instead of having increased by 3°C it will have increased by 2.82°C, which is of little relevance. In fact, the warming will merely be delayed by six years, reaching the temperature by 2106 instead. Hence, if the costs of the protocol (up to 4% of global GDP) are compared to the benefits of its implementation (a difference of 0.18°C depending on whether or not it is put into practice), the result is unconvincing.

Furthermore, if we take into account the fact that the fight against climate change does not only involve money, but that there are also costs in human lives, the figure

is also unconvincing. The number of deaths that can be attributed to climate change is relatively insignificant when compared to those caused by the diseases currently affecting the Third World. For example, deaths attributable to climate change are no more than 5% of those caused by AIDS, which suggests it would be better to dedicate the available resources to more pressing matters. According to the 'sceptical environmentalist' Bjørn Lomborg, just half the expenditure involved in the Kyoto Protocol (some 8 billion dollars), would make it possible to alleviate starvation around the world. Maybe starvation and birth control are the real problems of the Third World, and not (yet) climate change or respect for the environment. It is not in vain that many economists note that Africa does not require 'ecological' agriculture but agriculture full stop. What would be useful – they argue – would be to favour the transfer of non-polluting technologies to developing economies, while promoting energy saving measures, in addition to intelligent nuclear energy and renewable energy (hydro, wind, solar) in the developed world.

THE STATE OF THE ART

In its last five-year report, published in May 2007, the IPCC states: "It is 'likely' that there has been significant anthropogenic warming during the last 50 years in each continent, with the exception of Antarctica." It clarifies that it uses the term 'likely' to denote a probability of 67%, or rather that the statement in which it is used has a 33% probability of being false, a margin of error that should not be overlooked. Furthermore, by the end of this century, it predicts warming of around 3°C, and assures us that it is highly improbable that the figure will be below 1.5°C. It should be expected that this warming will persist for decades or centuries.

From the economy to politics

However, as we mentioned, economic factors are intricately linked to others, especially political ones. Hence, let us shift our attention from an economic analysis of Kyoto to a political one. What can we discover with this new perspective? Many things, some of which are extremely surprising…

First of all, we might ask why the majority of states (with the exception of the European Union, which we shall discuss later) do not wish to participate in the fight against climate change. The answer lies once again in mathematics. However,

this time it is not in chaos theory, but in game theory, specifically the so-called 'prisoner's dilemma'.

Game theory is a mathematical theory that attempts to explain how decisions are made in a hostile environment, of conflict, or rather when there are various 'deciders' (people who must make a decision) with opposing interests. The theory is extremely useful for making business decisions, but it can also be applied to political and even military decisions. It is not in vain that it was successfully applied to the study of various strategies in the nuclear arms race, the so-called 'war games'. Its creators include luminaries such as John von Neumann and John F. Nash (whose schizophrenia formed the basis of the award-winning film *A Beautiful Mind*).

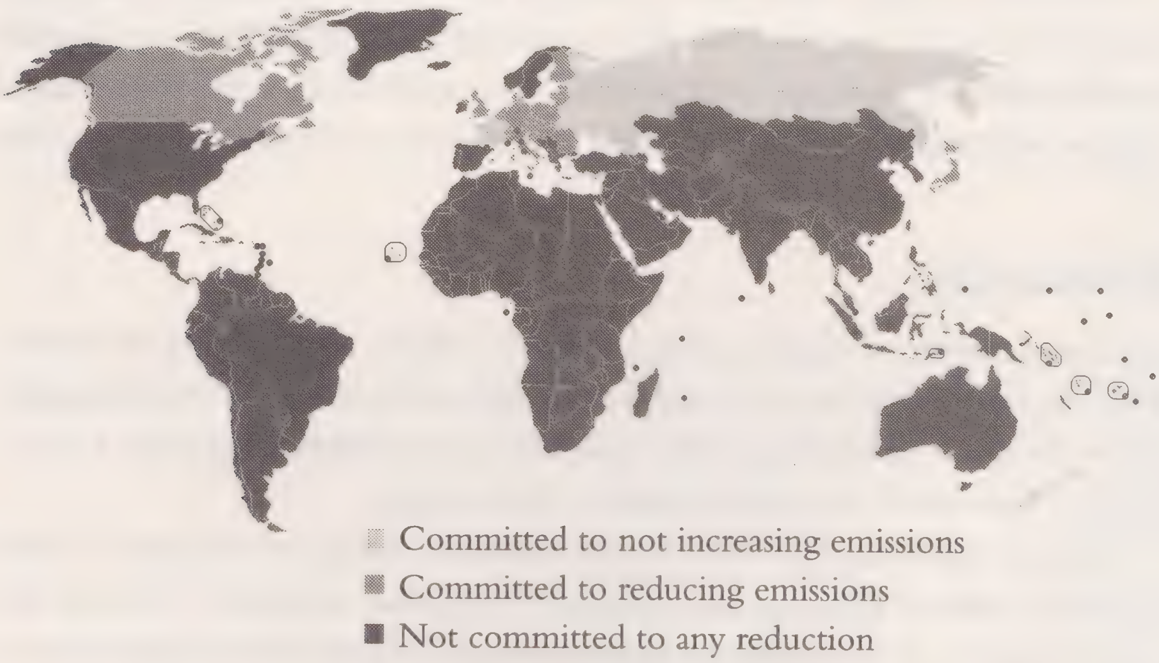
Formally stated by Albert W. Tucker, the prisoner's dilemma is a frequently occurring model of conflict: two criminals are arrested and locked up in cells so that neither can communicate with the other. The police suspect that they have taken part in a bank robbery, an offence punishable by up to 10 years in prison, but they have little evidence, meaning that they can only accuse the prisoners of the lesser offence of possession of illegal arms, punishable by two years in prison. However, if one of the thieves betrays his partner and collaborates with the police by providing evidence to incriminate their accomplice, their sentence will be halved (one year in prison), whereas the other will serve the full sentence for having robbed the bank (ten years in prison). Finally, if both confess and incriminate each other, the judge will only be able to sentence them to five years each.

There are two strategies or alternatives available to each prisoner: 'loyalty' or 'betrayal'. If the decision of the other prisoner is unknown, the safest strategy is always betrayal. If one of the prisoners remains loyal, but is betrayed by their accomplice, they risk ten years in jail, whereas if they betray them, in the worst-case scenario, they will only serve five years (when their accomplice also betrays them). However, this solution to the game is paradoxical in the sense that for both, the result of betraying each other (five years) is much better than if they choose to be loyal (one year).

Just like other environmental problems, climate change conforms to the prisoner's dilemma, since it creates an inefficient solution and a failure in the agreements. Let us consider an example. Imagine two countries *A* and *B* whose strategies with respect to the Kyoto Protocol can essentially be reduced to two options: to reduce emissions or not to reduce emissions. If *A* opts to reduce emissions and *B* does not, *A* will have more costs than benefits (because they will make all the effort), while *B* reaps the rewards of the situation and enjoys the reduction in CO_2 levels without incurring

costs (because air is a common good). Conversely, if *B* reduces emissions and *A* does not, *B* will lose out in financial terms with respect to *A*. Put this way, the best strategy for both *A* and *B* is not to cooperate, or in other words, not to comply with the Kyoto protocol, in spite of the fact that if both countries cooperate in the reduction of emissions, the benefit will be greater. The behaviour of each of the countries is rational, but the overall result of their decisions is not. We find ourselves once again facing the prisoner's dilemma.

Yet once again this leads us to another question: why do the European Union and the United States, the world's main producers of greenhouse gases who bear the greatest share of the economic burden of Kyoto, maintain diametrically opposite stances with respect to the protocol? Why does the European Union cooperate, in spite of the fact that game theory advises it not to? We could be forgiven for thinking that Europeans are much more aware of the environment and ecology than Americans. However, the answer is not so simple.



Map of the world showing the countries committed to reducing CO_2 emissions between 2008 and 2012 (26 November 2010).

The reality is that the dominant states in the European Union (Germany, France and the United Kingdom) are not required to make such a great effort as it might first seem in order to comply with Kyoto. In fact, since 1990, the base year of the

Protocol, many events have occurred on the geopolitical map. In Germany, with the fall of the Berlin Wall in 1989, the inefficient heavy industry of East Germany began to be dismantled due to the economic collapse, and hence emissions began to fall. In 1997, the emissions of the reunified Germany were already 12% below the base year for Kyoto (1990). In the case of France, in 1997 there was a stabilisation in emissions due to a change in the primary supply source of energy. In 2000 nuclear energy already represented two-fifths of France's total consumption. And finally, in 1997 the United Kingdom had already experienced a considerable reduction in emissions thanks to the energy policy of Margaret Thatcher, which consisted of replacing coal with natural gas, which by 2000 represented almost half of the British energy mix. Finally, the support shown to Kyoto by the countries of the former Soviet Union is explained by the fact that the dismantling of Soviet industry has resulted in a surplus of over 30% of CO_2 emissions, which these countries (Russia, Ukrainian, etc.) have begun to sell on the international carbon emissions market, obtaining lucrative earnings.

Finally, the United States has not ratified Kyoto and Australia is a latecomer since both countries make considerable use of coal. The United States is the world's second largest consumer, only exceeded by China, and Australia is the world's largest exporter, meaning that for these countries, the protocol is not a profitable investment.

The future(s)

The development of impoverished countries and the sustainability of wealthy countries will depend on our solution to a problem as complex as climate change. We are faced not with a single future, but with many possible alternatives. And our decisions will determine which of these becomes reality.

We must take great care of the environment because it is the system in which we live, but before adopting one environmental policy or another, we must also pay attention to all the available scientific evidence. Policies must pay attention to science, economics and mathematics, since putting inappropriate measures in place may be even more risky than doing nothing.

What makes an issue such as climate change so interesting is doubtless the fact that it requires everybody to work together: climatologists, meteorologists, physicists, mathematicians, economists, biologists, politicians... and even us, the public at large. And when it comes to this non-linear problem, chaos theory, the other main

character of this book, still has much to say. The theory has taught us two things. First that complex systems, such as the weather and the climate, have an underlying order, an internal structure, and second (which can be regarded as the opposite lesson), that simple systems can also have complex dynamics. In the end, when all is said and done, Friedrich Nietzsche will be proven right: "Everyone must organise the chaos they bear within."

Bibliography

GLEICK, J., *Chaos: making a new science*, New York, Viking Penguin, 1987.

KELLERT, S., *In the Wake of Chaos: unpredictable order in dynamic systems*, Chicago, University of Chicago Press, 1993.

LOMBORG, B., *The Sceptical Environmentalist. Measuring the real state of the world,* Cambridge, Cambridge University Press, 2001.

LORENZ, E., *The Essence of Chaos,* Washington, University of Washington Press, 1993.

RUELLE, D., *Chance and Chaos,* Princeton, Princeton University Press, 1991.

SMITH, P., *Explaining Chaos,* Cambridge, Cambridge University Press, 1998.

STEWART, I., *Does God Play Dice? The new mathematics of chaos*, London, Wiley–Blackwell, 2002.

Index

The Butterfly and the Tornado

Chaos theory and climate change

It is possible that readers will already be familiar with the hypothetical situation whereby a butterfly flapping its wings in Brazil can cause a tornado in Texas. However, the answer to the following question is perhaps more interesting: can a butterfly flapping its wings in Brazil PREVENT a tornado over Singapore? This book tackles this and other knotty problems related to the climate and its changes, as well as looking at other applications of chaos theory concerning populations and epidemics, brain signals and heartbeats.